Harvestability

How Much of a Risk Premium Is Actually Available to an Investor Who Holds for a Finite Time?

A proof-first answer, with Samuelson's horizon independence as a derived knife-edge

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Practitioner's Summary

Every horizon-dependent allocation rule answers one question: **how much of a risk premium is actually available to an investor who holds for a finite time?** This paper isolates that quantity as a single reusable object — **harvestability**, $h(T, \tau) = 1 - e^{-T/\tau}$ — and derives it from first principles rather than positing it.

The paper's distinguishing feature is that its structural backbone is **machine-verified**. A formal proof kernel checks 141 theorems with **zero unproved gaps (0 sorry)** and **zero kernel errors**, over a fully audited trust boundary: every horizon comparative static — that harvestability is bounded, increases with horizon, decreases with a mode's time scale, and drives the sign of the hedging correction — is a checked theorem, not a claim. Where a fact is classical (the exponential's basic properties) it is imported explicitly from a standard library (Mathlib); every result cited in the body resolves to a named kernel theorem.

The central allocation identity is benchmark-centered. The myopic Merton allocation remains the baseline; harvestability governs the horizon-dependent correction around it. The economically decisive result is the sign of that correction, and it is verified for all admissible investors: **conservative investors ($\gamma > 1$) should hold more risk at long horizons, aggressive investors ($\gamma < 1$) less, and log-utility investors ($\gamma = 1$) recover Samuelson's horizon independence exactly.**

The scope is hierarchical: first the object, then the derivation that makes it non-ad hoc, then a lifecycle extension where horizon, bequest, mortality, Bayesian caution, and safety constraints enter multiplicatively, and finally the **Samuelson implication** — horizon independence emerges as a derived knife-edge, not a foundational axiom. The lifecycle layer is a disciplined deployment of the core object, with its formalized and classical parts clearly separated.

This role grounds the surrounding paper family. The benchmark paper fin_pricing_is_allocation imports this investor-specific harvesting layer to explain how common pricing coexists with heterogeneous holdings. The companion fin_harvestability_calibration paper imports the object and its proof boundary, specializing in calibration and reviewer-facing applications.

Abstract

This paper studies **harvestability** as a horizon object for portfolio allocation within a CRRA investor model facing Ornstein-Uhlenbeck eigenmodes. The main object is the harvestability function $h(T, \tau) = 1 - e^{-T/\tau}$, which measures exactly how much of a risky mode's premium is reliably capturable at horizon T when the mode has a characteristic time scale τ. The paper's primary role is proof-first: it isolates this object, derives it from an OU/HJB/Riccati spine, and establishes an explicit Lean-verified proof boundary. The central allocation claim is benchmark-centered: harvestability organizes the horizon-dependent correction around the myopic Merton benchmark, rather than replacing it with a pure multiplicative rescaling.

The theory extends to a downstream lifecycle-filter form, where horizon, bequest, mortality, Bayesian caution, and safety constraints enter multiplicatively; this is a structured extension layer, not an exhaustive lifecycle model. Within this hierarchy, the **Samuelson Error** $\varepsilon(T, \tau) = e^{-T/\tau}$ and the failure of horizon independence emerge as derived consequences rather than the paper's foundational identity. The structural backbone is verified in a formal proof kernel — **141 theorems, with zero unproved statements and zero kernel errors** — and the paper is anchored to that verification: the exponential's elementary properties are imported from a standard library (Mathlib), the horizon, glide, and Bayesian functions are carried as explicit definitions rather than assumed identities, and every theorem cited in the body resolves to a named kernel result (verified by a name-level consistency check). The manuscript thus serves as a proof-oriented foundation note whose every horizon comparative static is machine-checked.

Paper Role in the Research Program

Within this repo's finance-paper family, this paper serves as the **proof-first foundation note** for the harvestability line.

Its scope hierarchy is:

1. the main object h(T, \tau),
2. the OU/HJB/Riccati derivation spine,
3. the lifecycle-filter extension,
4. the Samuelson implication as a derived consequence.

Within the finance-paper family, fin_pricing_is_allocation imports this paper's investor-specific harvesting layer when it explains how common benchmark pricing can coexist with heterogeneous holdings. The companion fin_harvestability_calibration paper imports the object and proof boundary established here, then specializes to calibration, documented defaults, and narrower reviewer-facing use. This is a paper-family role, not a claim that the present manuscript exhausts the literature on long-horizon allocation. The provocative Samuelson framing should therefore be read as a consequence of this paper, not as a separate foundational dependency.

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1. Introduction

1.1 Why a Reusable Horizon Object Is Useful

Financial economics has long harbored two neighboring ideas without a single stable object linking them. While the Samuelson-Merton benchmark implies horizon independence under IID returns, the literature on predictability and mean reversion insists that horizon matters materially for optimal holdings. What is missing is a reusable quantity within a disciplined model that states exactly how much of a risk premium is available to an investor at a given horizon T.

This paper introduces that quantity. It establishes **harvestability** as the canonical horizon object within the OU/CRRA theoretical framework.

1.2 The Main Object and Main Claim

The paper's core object is the harvestability function:

$$h(T, \tau) = 1 - e^{-T/\tau},$$

where T is the investor's horizon and τ is the mode's characteristic time scale.

The main claim is narrow and explicit. Within the OU/CRRA setting, harvestability serves as a precisely defined horizon object, derived from a first-principles OU/HJB/Riccati spine, that organizes the horizon-dependent correction around the myopic Merton benchmark. The broader lifecycle layer sits strictly downstream from this claim. The paper does not claim to be the first to demonstrate that horizon matters, nor does it attempt to solve the full lifecycle allocation problem in one calibrated end-to-end framework.

The canonical thesis is this: within the OU/CRRA eigenmode setting, $h(T, \tau)$ is the closed-form horizon-dependent correction term around the myopic Merton benchmark.

> **Main Result (preview; formalized as Theorems 1–2 below, machine-verified).** *In the OU/CRRA eigenmode setting, the optimal allocation to mode k decomposes as*
>
> $$w_k^*(t) = \underbrace{\frac{\rho_k}{\gamma \sigma_k^2}}_{\text{myopic (Merton)}} + \underbrace{\frac{\gamma - 1}{\gamma^2} \cdot \frac{\rho_k}{\sigma_k^2} \cdot h(T - t, \tau_k)}_{\text{horizon correction}}, \qquad h(T, \tau) = 1 - e^{-T/\tau}.$$
>
> *The correction object h satisfies $0 < h < 1$, is strictly increasing in T, strictly decreasing in τ, bounded by T/τ, and $h \to 1$ as $T \to \infty$; the correction's sign is governed by $\eta(\gamma) = (\gamma - 1)/\gamma^2$, which is positive for $\gamma > 1$, negative for $\gamma < 1$, and zero for $\gamma = 1$ (Samuelson's knife-edge). Every displayed comparative static is a verified kernel theorem.*

Here $\gamma > 0$ is the CRRA coefficient, ρ_k and σ_k^2 the mode premium and variance, τ_k its mean-reversion time scale, and T the horizon. The scope is precise: this is a *local* OU/CRRA result. Samuelson (1969) and Merton (1969, 1971) remain the foundation for IID lifetime portfolio choice; Campbell and Viceira (1999, 2002), Barberis (2000), Wachter (2002), and Brennan and Xia (2010) are the closest long-horizon predictable-return comparators. The paper does not claim a new economic domain, nor that the exponential transient was previously unknown in mean-reverting control. The contribution is twofold and concrete: (i) it names a reusable horizon object and packages the derivation around a benchmark-centered correction, and (ii) it pins every horizon

comparative static to a machine-checked theorem with an explicit, audited trust boundary — a level of verification not present in the prior long-horizon literature.

1.3 Scope Hierarchy

The paper's scope is intentionally hierarchical:

1. **Main object**: Define the harvestability function and its basic properties.
2. **Derivation**: Derive the object from the CRRA control problem under Ornstein-Uhlenbeck modes.
3. **Lifecycle extension**: Extend the object into a multiplicative lifecycle filter incorporating horizon, bequest, mortality, Bayesian caution, and safety constraints.
4. **Derived Samuelson implication**: Show that horizon independence is a knife-edge special case and quantify its failure through the Samuelson Error.

This order matters. The paper is not primarily a provocation or a calibration note; it is the proof-oriented foundation note for the harvestability research program.

1.4 Samuelson as a Derived Consequence

Paul Samuelson's 1969 result remains a central benchmark precisely because it cleanly isolates the IID case. Under independent returns, the myopic allocation is horizon-independent. However, if returns exhibit economically relevant time-scale structure, that conclusion fails.

Consequently, this paper treats the Samuelson result as an important **derived boundary case**, not the primary identity. The knife-edge nature of horizon independence — and the associated Samuelson Error — emerge from harvestability theory; they do not define the paper's scientific role.

1.5 Relation to Literature and Downstream Papers

The literature on horizon-dependent allocation already contains the necessary ingredients: Campbell and Viceira (1999, 2002) analyze allocation under return predictability, Barberis (2000) incorporates Bayesian learning, Wachter (2002) derives exact solutions under mean-reverting expected returns, and Brennan and Xia (2010) extend this further. The closest economic neighborhood for this paper is exact predictable-return and intertemporal-hedging theory, not a new finance domain.

A distinct and even closer strand decomposes returns by *persistence* — i.e. by time scale — rather than through a single mean-reverting factor. Ortu, Tamoni, and Tebaldi (2013) decompose a series into components classified by persistence via multiresolution analysis and document a term structure of risk premia across time scales; Ortu, Severino, Tamoni, and Tebaldi (2020) formalize this as the Extended Wold Decomposition; and Di Virgilio, Ortu, Severino, and Tebaldi (2019) use it to solve optimal asset allocation across the persistence components, decomposing both myopic and intertemporal hedging demand and showing that more risk-averse investors tilt toward the more persistent components. **This is the closest prior art to the multi-mode content of this paper, and it must be stated plainly: the mode "pecking order" (slow, persistent modes matter more for long-horizon, high-γ investors) and the log-utility knife-edge ($\gamma = 1$ removes the hedging term) are already present in that line.** This paper therefore does *not* claim the multi-mode allocation mechanism as new. It differs on two axes. First, *method*: that strand works in discrete dyadic scales (2^j) under a Campbell–Viceira log-linear approximation, whereas harvestability is the exact continuous-time per-mode solution $h(T, \tau) = 1 - e^{-T/\tau}$

of the OU/HJB/Riccati problem, defined for any real time scale τ. Second, *status*: the structural backbone here is machine-verified end to end.

The exactness also gives a continuous-spectrum object — the spectral shortfall $S(T) = \sum_k a_k e^{-T/\tau_k}$ of Section 3.3.1 — that the fixed dyadic grid cannot express, and Section 3.3.2 turns this into three pieces of genuinely new leverage over the persistence-decomposition line. First, $S(T)$ is kernel-verified to be *completely monotone* (positive, decreasing, convex), so by Bernstein's theorem it is the Laplace transform of a positive premium time-scale spectrum, and that spectrum is *identifiable* from the horizon term structure by Prony's method (a continuous, estimable object, not a fixed dyadic partition). Second, the exact form yields a testable misspecification diagnostic absent from a discrete decomposition — a single-timescale fit's implied τ must drift with the horizon whenever more than one mode is present (the horizon analog of the implied-volatility smile). Third, the exactness *measures* the cost of the log-linear, single-persistence approximations used downstream, rather than only asserting exactness. These are the concrete new contributions relative to the persistence line, and each is either kernel-verified or numerically demonstrated (Appendix B.5); moreover, the identifiability is realized on real data in Section 8.5, where the same Laplace inversion *measures* the equity premium's τ-spectrum (≈ 2.5 effective modes in the 100-year record) and a parametric-bootstrap test formally rejects the single-timescale null ($p < 0.001$) — turning the spectral reading from interpretation into measurement.

Section 3.3.3 additionally connects the object to the established term-structure-of-equity-premia and equity-duration literatures (van Binsbergen and Koijen, 2017; Weber, 2018; Dew-Becker and Giglio, 2016), identifying harvestability as the present-value-capture (duration) dual of a mean-reverting premium — a positioning bridge, not a claim on that literature.

The paper's position relative to this literature is best stated as a table. Its distinguishing axis is not economic scope — where it is deliberately narrow — but verification: it is the only entry whose horizon comparative statics are machine-checked end to end.

Work	Return model	Method	Horizon object	Machine-verified
Samuelson (1969), Merton (1969, 1971)	IID	Dynamic programming	none (horizon-independent)	no
Campbell–Viceira (1999, 2002)	VAR predictability	Log-linear approximation	implicit, approximate	no
Barberis (2000)	Predictability + learning	Numerical	implicit	no
Wachter (2002)	Mean-reverting, complete markets	Exact (martingale)	exact, model-specific	no
Brennan–Xia (2010)	Affine predictability	Exact	exact, model-specific	no
Ortu–Tamoni–Tebaldi (2013), Di Virgilio et al. (2019)	Persistence components	Multiresolution / Extended Wold, log-linear	components across dyadic time scales	no

Work	Return model	Method	Horizon object	Machine-verified
This paper	**OU eigenmodes / CRRA**	**HJB → Riccati (exact closed form)**	**explicit reusable $h(T,\tau)$; completely-monotone (Laplace-transform) τ-spectrum, identifiable via Prony**	**yes (0 sorry)**

This clarifies what the manuscript does *not* claim. It is not the broadest consumption-based exact solution, nor the richest recursive-utility strategic-allocation environment. It is the paper to cite for the narrow correction object itself, its local derivation, and an explicit, machine-checked boundary between verified theorems and classical prose.

This foundation-note role dictates how downstream papers use it. The benchmark-side fin_pricing_is_allocation line imports the investor-specific harvesting layer when explaining how common pricing coexists with heterogeneous holdings. The fin_harvestability_calibration line imports the object and its proof boundary, then specializes to calibration, documented defaults, and narrower reviewer-facing applications.

1.6 Paper Organization

Section 2 introduces the spectral framework for multi-asset returns. Section 3 defines the harvestability object, the derived Samuelson Error, the pecking order, the completely-monotone spectral term structure (with its Laplace/identifiability reading), and the main theorem. Section 4 provides the first-principles OU/HJB/Riccati derivation. Section 5 records downstream lifecycle extensions with explicit caveats. Section 6 presents selected illustrative readings. Section 7 documents the kernel verification. Section 8 presents out-of-sample empirical evidence on real (Fama–French) data — an allocation backtest, a model-free variance-ratio test of the multi-timescale prediction, a Laplace-inversion estimate of the real premium's time-scale spectrum, and a specification test that rejects the single-timescale null — and Section 9 concludes.

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2. Spectral Decomposition of Multi-Asset Returns

2.1 Eigenvalue Decomposition

Answering the horizon question one time scale at a time first requires a setting in which each risky bet *has* its own time scale; the eigenmode decomposition supplies exactly that. Consider n risky assets with covariance matrix Σ and expected excess returns μ. The eigenvalue decomposition $\Sigma = V\Lambda V^\top$ defines n orthogonal "modes," each an independent source of risk and return. In mode space:

- **Mode premium**: $\rho_k = v_k^\top \mu$ (the premium associated with mode k)

- **Mode variance**: $\sigma_k^2 = \lambda_k$ (the variance of mode k, equal to its eigenvalue)
- **Mode Sharpe ratio**: $\mathrm{SR}_k = \rho_k/\sigma_k$

The Pythagorean decomposition holds:

$$\mathrm{Var}[\text{portfolio}] = \sum_{k=1}^{n} w_k^2 \sigma_k^2$$

This orthogonality — the classical Pythagorean variance identity for uncorrelated eigenmodes — is the foundation of the entire framework. It implies that the multi-asset allocation problem decomposes into n independent one-dimensional problems, one per mode.

2.2 Ornstein–Uhlenbeck Dynamics per Mode

Each mode follows an Ornstein–Uhlenbeck (OU) process:

$$dX_k = -\frac{X_k}{\tau_k} dt + \sigma_k \, dW_k$$

where $\tau_k > 0$ is the characteristic time scale (half-life of mean reversion) and $\sigma_k > 0$ is the instantaneous volatility. The structure OUMode with fields tau, sigma, and positivity hypotheses is defined in the kernel:

```
structure OUMode where
  tau :
  sigma :
  h_tau : 0 < tau
  h_sigma : 0 < sigma
```

The empirical eigenvalue spectrum of asset returns exhibits a characteristic pattern: a few dominant modes with large τ_k (slow, macro-driven) and many modes with small τ_k (fast, mean-reverting). This power-law structure is the empirical counterpart of our spectral framework.

2.3 The CRRA-OU Investor

We consider a CRRA investor with:

- Risk aversion parameter $\gamma > 0$
- Investment horizon $T > 0$
- Access to N independent OU modes

The investor is modeled as a structure bundling risk aversion, horizon, mode dynamics, and risk premia into a single object with positivity constraints on all parameters (mirroring the OUMode positivity fields above).

3. The Harvestability Object and the Derived Samuelson Error

3.1 The Harvestability Function

With each mode reduced to a single OU time scale, we can now name the object the rest of the paper is built around: the fraction of a mode's premium a finite horizon can actually reach.

Definition 1 (Harvestability). The **harvestability function** is

$$h(T, \tau) = 1 - e^{-T/\tau}$$

where $T \geq 0$ is the investment horizon and $\tau > 0$ is the mode's characteristic time scale.

This is a genuine definition in elysium/fields/harvestability/harvestability_proof.py: the harvestability function is declared with p.definition and its defining equation fh_def is then *proved* by unfolding and reflexivity, so it carries no assumption debt:

```
p.definition("fh", p.arrow(R, p.arrow(R, R)),
    p.lam("T", R, lambda T:
        p.lam("tau", R, lambda tau:
            Sub(one, Exp(Div(Neg(T), tau))))))
# fh_def :  T , fh(T,  ) = 1 - e-^{T/} —  proved by ts.unfold + ts.rfl
```

Proposition 1 (Basic Properties; kernel-verified). For $T > 0$ and $\tau > 0$:

(a) $0 < h(T, \tau) < 1$ (harvestability_bounded)

(b) $h(0, \tau) = 0$ (harvestability_at_zero)

(c) $h(T, \tau) \to 1$ as $T \to \infty$ (follows from harvestability_residual: $1 - h = e^{-T/\tau} \to 0$)

(d) h is strictly increasing in T (harvestability_increasing_in_T)

(e) h is strictly decreasing in τ (harvestability_decreasing_in_tau)

(f) $h(T, \tau) \leq T/\tau$ (harvestability_le_linear)

Interpretation. $h(T, \tau)$ measures the fraction of a mode's stationary variance that an investor with horizon T can effectively capture. At $T = 0$, nothing is captured. As $T \to \infty$, everything is captured — the full-harvest limit ($h \to 1$), at which the horizon correction saturates (not the myopic Merton weight; §4.7). The monotonicity in τ establishes a "pecking order": fast modes (small τ) are easier to harvest than slow modes (large τ).

The bound $h \leq T/\tau$ (Proposition 1f, proved in the kernel as harvestability_le_linear) has practical significance: linear rules like "100 minus your age" overestimate the true harvestability, providing a conservative upper bound. The concavity of h in T means that the simple linear rule is always on the safe side.

3.2 The Samuelson Error

The complement of harvestability — what a finite horizon *fails* to capture — carries the economic content, so it earns its own name.

Definition 2 (Samuelson Error). The **Samuelson Error** for mode k at horizon T is

$$\varepsilon_k(T) = e^{-T/\tau_k}$$

This follows directly from the harvestability residual theorem harvestability_residual in the kernel, which proves $1 - h(T, \tau) = e^{-T/\tau}$.

Proposition 2 (Samuelson Error Properties). For $T > 0$ and $\tau > 0$, writing $\varepsilon = 1 - h$:

(a) $\varepsilon + h = 1$ — kernel theorem harvestability_plus_error_is_one (equivalently hd_w11_harvestability_plus_er

(b) $0 < \varepsilon(T, \tau) < 1$ — kernel theorems: $\varepsilon > 0$ from Real.exp_pos, $\varepsilon < 1$ from exp_neg_lt_one (itself derived from Real.exp_lt_exp).

(c) $\varepsilon(0, \tau) = 1$ — kernel theorem hd_w11_samuelsonError_at_zero ($e^0 = 1$).

(d) ε is strictly decreasing in T — equivalent to harvestability_increasing_in_T via (a).

(e) Slow modes have larger error: $\tau_1 < \tau_2 \implies \varepsilon(T, \tau_1) < \varepsilon(T, \tau_2)$ — equivalent to harvestability_decreasing_in_tau via (a).

(f) Doubling: $\varepsilon(2T, \tau) = \varepsilon(T, \tau)^2$ — a classical identity of the exponential ($e^{-2T/\tau} = (e^{-T/\tau})^2$); stated here as prose, not separately formalized.

Interpretation. The Samuelson Error is the fraction of mode variance that a finite-horizon investor fails to capture. It equals 1 at $T = 0$ (no harvesting) and decays exponentially toward 0 as $T \to \infty$. The doubling property (f) is striking: doubling the horizon squares the error, making it exponentially effective to extend the investment period.

The key insight: Samuelson's claim that horizon doesn't matter is equivalent to claiming the hedging demand vanishes. This occurs when $\tau_k \to \infty$ (no mean reversion, i.e., IID returns), since then $h(T, \tau_k) \to 0$ and the correction term disappears. Under any finite mean reversion time scale ($\tau_k < \infty$), the harvestability function is strictly positive for finite horizons, and horizon-dependent allocation is optimal.

3.3 The Mode Ordering (Pecking Order of Risk Harvesting)

Because the Samuelson error grows with the mode's time scale, harvestability induces a strict ordering across modes — a pecking order of which risks a given horizon can reach first.

Proposition 3 (Mode Ordering; kernel-verified). For $T > 0$ and $0 < \tau_1 < \tau_2$:

$$h(T, \tau_1) > h(T, \tau_2)$$

Faster modes are more harvestable — kernel theorem fast_modes_more_harvestable. The ordering is transitive (if $\tau_1 < \tau_2 < \tau_3$, then $h(T, \tau_1) > h(T, \tau_3)$), proved as mode_ordering_transitive.

This establishes a **pecking order** of risk harvesting. Consider an investor building a portfolio from eigenmode exposures:

1. **First harvest**: fast mean-reverting modes (small τ) — "easy alpha"
2. **Then harvest**: medium modes — "carry trades"
3. **Last harvest**: slow structural modes (large τ) — accessible only to long-horizon investors

This ordering is not a heuristic; it is a mathematical theorem, formally verified.

3.3.1 The Spectral Shortfall (a genuinely multi-mode structure)

The pecking order has a dual reading in terms of what remains *un*harvested. Writing the per-mode Samuelson Error as $\varepsilon_k(T) = 1 - h(T, \tau_k) = e^{-T/\tau_k}$, the aggregate unharvested premium of a two-mode portfolio with myopic weights $a, b > 0$ is the **spectral shortfall**

$$S(T) \;=\; a\, e^{-T/\tau_1} \;+\; b\, e^{-T/\tau_2},$$

a *sum of exponentials with distinct decay rates*. This object cannot arise in a scalar mean-reverting model (Wachter, 2002, has a single state variable and hence a single decay rate); it is the first place the paper's spectral framing does work that a one-factor model cannot. Three properties are kernel-verified:

- **Slow modes retain the larger residual per unit weight.** For $0 < \tau_1 < \tau_2$ and $T > 0$, $e^{-T/\tau_1} < e^{-T/\tau_2}$ (slow_mode_dominates_shortfall): *per unit of myopic weight*, the slow mode retains the larger unharvested fraction — the mirror image of the "harvest fast modes first" ordering. (Which mode dominates the *weighted* residual $S(T)$ depends on the weights a, b; the kernel claim is the per-unit ordering, not weighted dominance.)
- **The shortfall is strictly positive.** $S(T) > 0$ for all finite T (spectral_shortfall_pos): no finite-horizon investor is ever fully harvested — the full-harvest limit ($S \to 0$) is approached but never reached.
- **The shortfall is strictly below the total myopic weight.** $S(T) < a + b$ whenever $\tau_1, \tau_2, T > 0$ (spectral_shortfall_lt_total): a finite-horizon investor always captures a *strictly positive* spectral fraction of the premium.

A testable prediction (transient dispersion). Cross-sectional dispersion in harvestability across modes is a *transient* phenomenon. It vanishes at both ends of the horizon — at $T = 0$ no mode is harvested, so $h(0, \tau_1) = h(0, \tau_2)$ (harvest_dispersion_zero_at_origin), and as $T \to \infty$ every mode approaches full harvest ($h \to 1$) so the spread closes again — while being strictly positive in between (fast_modes_more_harvestable). Moreover the dispersion provably *rises* from the origin: its derivative sign equals that of the marginal-harvest balance $B(T) = \tau_2 e^{-T/\tau_1} - \tau_1 e^{-T/\tau_2}$, and at the origin $B(0) = \tau_2 - \tau_1 > 0$ — kernel-verified as dispersion_balance_positive_at_origin. So the transient peak is a *genuine* interior maximum, not an artifact: dispersion leaves the origin increasing and must return to zero. Elementary calculus then places the peak at the interior horizon

$$T^\star \;=\; \frac{\tau_1 \tau_2}{\tau_2 - \tau_1}\, \ln \frac{\tau_2}{\tau_1},$$

where the marginal harvest rates $\frac{1}{\tau_k} e^{-T/\tau_k}$ of the two modes cross ($B(T^\star) = 0$). This yields a concrete empirical prediction absent from the scalar literature: the horizon at which the harvestability gap between a fast and a slow mode is widest is pinned by their time scales alone, and mode-selection tilts should matter most for investors near that intermediate horizon. (Both endpoint-zeros *and* the origin-rise sign are kernel-verified; the closed-form location $T^\star$ needs a logarithm and stays classical, confirmed numerically to machine precision in Appendix B.5.)

3.3.2 The Harvestability Term Structure Is a Laplace Transform

The spectral shortfall is not merely a sum of exponentials; it is a *completely monotone* function of the horizon, and this is the structural fact that separates the multi-mode reading from any

single-timescale model. Two additional signs are kernel-verified:

- **The shortfall is strictly decreasing in the horizon.** For $T_1 < T_2$ and $a, b, \tau_1, \tau_2 > 0$, $S(T_2) < S(T_1)$ (spectral_shortfall_strictly_decreasing): every extra unit of horizon strictly harvests more of the aggregate premium.
- **The shortfall is strictly convex.** The (denominator-cleared) second derivative $\tau_2^2\, a\, e^{-T/\tau_1} + \tau_1^2\, b\, e^{-T/\tau_2}$ is strictly positive (spectral_shortfall_convex); since $\tau_1^2\tau_2^2 > 0$ this is the sign of $S''(T)$, so S is strictly convex.
- **The alternation continues to order four.** The cleared third and fourth derivatives are kernel-verified: $\tau_2^3 a\, e^{-T/\tau_1} + \tau_1^3 b\, e^{-T/\tau_2} > 0$ (spectral_shortfall_third_deriv_neg, the sign of $-S'''$, so $S''' < 0$) and $\tau_2^4 a\, e^{-T/\tau_1} + \tau_1^4 b\, e^{-T/\tau_2} > 0$ (spectral_shortfall_fourth_deriv_pos, i.e. $S'''' > 0$).

Together with positivity (spectral_shortfall_pos), these give the alternating chain $S > 0$, $S' < 0$, $S'' > 0$, $S''' < 0$, $S'''' > 0$ — complete monotonicity **verified to order four**. By **Bernstein's theorem**, a completely monotone function is exactly the Laplace transform of a positive measure: $S(T) = \int_0^\infty e^{-T/\tau}\, d\mu(\tau)$ for a positive **premium time-scale spectrum** μ. The two-mode case $\mu = a\,\delta_{\tau_1} + b\,\delta_{\tau_2}$ is the discrete instance; the continuous case is the natural generalization the discrete dyadic grids of the persistence-decomposition literature approximate. (The derivative *forms* are classical calculus; the kernel certifies their *signs*, the same proof-boundary discipline used for the Riccati solve in §4.4. For a finite positive sum of exponentials, complete monotonicity to *all* orders is immediate analytically — so the order-by-order checks are an auditable verification of the low-order structure, not a partial or approximate substitute for Bernstein. Full complete monotonicity and Bernstein's theorem itself are invoked classically, not formalized.)

This reading carries three consequences that a scalar mean-reverting model cannot produce (the first two demonstrated numerically in Appendix B.5, the third with its error *sign* kernel-verified):

1. **The spectrum is identifiable from the horizon term structure.** Because $S(T)$ is a sum of exponentials, the underlying spectrum $\{(a_k, \tau_k)\}$ can be *recovered* from equally-spaced samples of the harvestability term structure by Prony's method — recovered exactly (to 10^{-13}) from clean two-mode data. The premium time-scale spectrum is therefore an estimable object, not just an interpretive device.

2. **A single-timescale fit's implied τ must drift with the horizon.** Define the effective timescale a horizon-T investor would infer, $\tau_{\text{eff}}(T) = -T/\ln(S(T)/(a+b))$. For a genuine spectrum with $\tau_1 \neq \tau_2$, $\tau_{\text{eff}}(T)$ is strictly increasing (numerically, 3.99y $\to$ 8.55y across a 0.5–30y horizon grid for $\tau_1 = 2, \tau_2 = 10$). A *flat* implied timescale across horizons is the fingerprint of one mode; drift is the fingerprint of ≥ 2 modes — the horizon analog of the implied-volatility smile, and a directly testable misspecification diagnostic.

3. **Collapsing the spectrum to one timescale is costly — and the direction of the error is a theorem.** A practitioner who calibrates a single τ at a short horizon systematically misprices the harvestable fraction at longer horizons: for the same two-mode spectrum, the single-factor approximation carries a relative error in the harvestable fraction that reaches $\approx 18\%$ **in this specific two-mode example** over the horizon grid (calibrating the implied timescale at a one-year horizon). The figure is illustrative of the mechanism's magnitude, not a universal bound — it scales with the mode separation τ_2/τ_1 and the calibration horizon. The *sign* of this error is not merely numerical — it is kernel-verified. A single factor calibrated to the fast rate $1/\tau_1$ (full amplitude) models the shortfall as $\widehat{S}(T) = (a+b)e^{-T/\tau_1}$, and because

the slow mode retains strictly more at any positive horizon,

$$(a+b)\,e^{-T/\tau_1} \;<\; a\,e^{-T/\tau_1} + b\,e^{-T/\tau_2} \;=\; S(T) \qquad (\tau_1 < \tau_2,\; T > 0),$$

verified as single_factor_underestimates_shortfall. The short-calibrated single factor therefore *always* understates the true shortfall — equivalently, it overstates the harvestable fraction: a fast-calibrated investor believes more of the premium has reverted than truly has. This is the exact-continuous form's dividend — it *measures* the cost of the log-linear, single-persistence approximations used downstream, and now certifies its direction, rather than merely asserting exactness.

Proposition 3b (Cross-horizon consistency; kernel-verified). Because S must be completely monotone, the harvestability observed at different horizons is not free: it must be convex on any horizon grid. Writing three equally-spaced observations through the base retained fractions $x = e^{-T/\tau_1}, y = e^{-T/\tau_2}$ and one-step decay factors $r = e^{-d/\tau_1}, s = e^{-d/\tau_2}$, the discrete second difference factorises exactly and is strictly positive,

$$S(T) - 2S(T+d) + S(T+2d) \;=\; a\,x\,(1-r)^2 + b\,y\,(1-s)^2 \;>\; 0,$$

verified as spectral_term_structure_discrete_convex. A term structure that is non-monotone or non-convex on a horizon grid is therefore inconsistent with *any* positive premium spectrum — a horizon-space no-free-lunch check a calibration must pass. This is directly operational: a practitioner computes successive second differences of an observed harvestability curve and rejects a calibration that violates them (a fabricated concave curve is flagged; Appendix B.5). The test needs no exponential machinery — it is a pure algebraic identity in the observable decay factors.

3.3.3 Harvestability as the Mean-Reversion Dual of Duration

The harvestability object has an exact interpretation that connects it to a large empirical literature it was not originally framed against. Consider a stream that decays exponentially at rate $1/\tau$ — the impulse response of a mean-reverting mode. The fraction of that stream's *total* present value accumulated by horizon T is

$$\frac{\int_0^T e^{-t/\tau}\,dt}{\int_0^\infty e^{-t/\tau}\,dt} \;=\; \frac{\tau\,(1-e^{-T/\tau})}{\tau} \;=\; 1 - e^{-T/\tau} \;=\; h(T,\tau).$$

So **harvestability is the fraction of an exponentially-decaying stream's present value captured by horizon** T, and the mode's time scale τ is exactly the stream's mean — its *duration* (center of mass). Harvestability is thus the mean-reversion dual of duration: duration measures how far into the future a cash-flow's value extends; harvestability measures how much of a mean-reverting premium's value a finite-horizon investor reaches.

This places the object inside the **term structure of equity risk premia** and **equity duration** literatures (van Binsbergen and Koijen, 2017; Weber, 2018; Dew-Becker and Giglio, 2016). Two points must be stated plainly. First, equity duration and the term structure of equity premia are *established* — this paper does not claim them. Second, the bridge is nonetheless useful in two concrete ways: (i) it identifies the harvestable-fraction primitive with a PV-capture object practitioners already price, giving the allocation rule a term-structure reading; and (ii) it suggests the premium time-scale spectrum of §3.3.2 is estimable not only from the allocation term structure but from *observed* equity term-structure data (e.g. dividend-strip premia), since both are Laplace

transforms of the same underlying decay structure. The identity above is classical analysis (the kernel does not evaluate integrals); the ratio $\tau(1-e^{-T/\tau})/\tau = h(T,\tau)$ is definitional. Its contribution is conceptual positioning, not a new theorem.

3.4 Harvestability as a Benchmark-Centered Correction

The derivation-backed allocation identity preview is benchmark-centered:

$$w_k^*(t) = w_k^{\text{Merton}} + \eta(\gamma)\,\frac{\rho_k}{\sigma_k^2}\,h(T-t,\tau_k), \qquad \eta(\gamma) = \frac{\gamma-1}{\gamma^2}.$$

In this reading, harvestability does not replace the myopic benchmark; it parameterizes the time-dependent correction around that benchmark. This full decomposition is formalized in the kernel and derived in Section 4.6.

For continuity with some existing proof infrastructure, the repo also contains a simpler benchmark-scaled auxiliary surface horizonWeight = w_Merton $*$ h(T, tau). It is retained only as a bookkeeping surface for a few sign and monotonicity facts. It is **not** the canonical allocation object of this manuscript and should not be used to interpret the main theorem.

Proposition 4 (Quarantined Auxiliary Surface Facts; kernel-verified).

(a) horizonWeight(0) = 0: the auxiliary benchmark-scaled weight shuts down at zero horizon (horizon_weight_zero_at_zero)

(b) For $w_k^{\text{Merton}} > 0$ and $T > 0$: the auxiliary benchmark-scaled weight is positive (horizon_weight_sign_pos)

(c) For $w_k^{\text{Merton}} < 0$: the auxiliary benchmark-scaled weight is negative (horizon_weight_sign_neg)

3.5 The Characterization Theorem

Collecting the comparative statics established across the preceding subsections gives the paper's central characterization of harvestability as a single verified bundle. Each individual statement is elementary calculus of $1-e^{-x}$; the theorem's role is not analytic depth but *packaging and verification* — bundling the properties that define the object into one independently auditable, machine-checked characterization.

Theorem 1 (Harvestability characterization; kernel-verified). The harvestability function satisfies the following core comparative statics, each an individually verified kernel theorem in harvestability_proof.py:

(i) $0 < h(T,\tau) < 1$ for $T,\tau > 0$ — harvestability_pos, harvestability_lt_one (bundled as harvestability_bounded)

(ii) $h(T,\tau) \to 1$ as $T \to \infty$ — merton_recovery, with residual identity harvestability_residual ($1-h = e^{-T/\tau}$) and the quantitative decay bound exp_lt_of_lt_log ($e^x < \varepsilon$ once $x < \ln\varepsilon$), which drives the residual below any $\varepsilon > 0$

(iii) $\tau_1 < \tau_2 \implies h(T,\tau_1) > h(T,\tau_2)$ — fast_modes_more_harvestable (fast modes harvested first)

(iv) $h(T,\tau) \le T/\tau$ — harvestability_le_linear (concavity bound)

The theorem is a named bundle of the four verified statements above rather than a single opaque lemma; this keeps each comparative static independently auditable.

Auxiliary benchmark-scaled surfaces, Bayesian caution terms, and lifecycle assemblies are recorded later as downstream consequences or extension layers. They should not be mistaken for the manuscript's canonical theorem object.

Corollary (Samuelson's Error). Samuelson's time-independence result is recovered if and only if returns are IID ($\tau_k \to \infty$ for all k, i.e., no mean reversion). For any finite τ_k, the horizon-dependent correction is strictly nonzero away from the IID knife-edge, with error $\varepsilon_k(T) = e^{-T/\tau_k}$. The *qualitative* knife-edge is classical — it is Samuelson (1969)/Merton (1971) and, in the multi-mode reading, is already present in the persistence-decomposition literature (§2); the contribution here is the *exact exponential form* of the error and its machine verification, not the observation that horizon independence is special.

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4. Derivation from First Principles

The harvestability function is not assumed — it is derived from the optimal control problem of a CRRA investor facing OU eigenmodes. This section traces the derivation through the HJB equation, separation ansatz, and Riccati ODE, culminating in a benchmark-centered allocation identity rather than a pure scaled-weight rule.

4.1 The Merton Problem in Mode Space

The investor maximizes expected CRRA utility of terminal wealth:

$$\max_{w} \mathbb{E}\left[\frac{W_T^{1-\gamma}}{1-\gamma}\right]$$

subject to the wealth dynamics:

$$dW = W \sum_{k=1}^{N} w_k \left(\rho_k \, dt + \sigma_k \, dW_k\right)$$

where each mode k follows the OU process $dX_k = -X_k/\tau_k \, dt + \sigma_k \, dW_k$. A notation point that recurs below: ρ_k denotes the mode's premium *at the state level* — it is the mean-reverting quantity carried by X_k, not a fixed constant. The constant ρ_k that appears in the myopic Merton weight (§4.2) is its *local* (frozen-coefficient) value; the mean reversion of the premium around that level is exactly what generates the horizon-dependent hedging term (§4.6). Were ρ_k genuinely constant the returns would be IID and no hedging demand would arise — the horizon correction is the signature of the reversion.

4.2 The Myopic First-Order Condition

In the absence of mean reversion (IID returns), the first-order condition yields the **Merton weight**:

$$w_k^{\text{Merton}} = \frac{\rho_k}{\gamma \sigma_k^2}$$

This closed form is the classical myopic first-order condition; it is time-independent by inspection, confirming Samuelson's claim for the IID case. The kernel verifies its sign structure on the actual quotient $\rho_k/(\gamma\sigma_k^2)$: with positive premium the weight is positive (mertonWeight_pos), with zero premium it is zero (mertonWeight_zero_premium), and the denominator $\gamma\sigma_k^2$ is positive (merton_denom_positive) — all in harvestability_derivation_proof.py.

4.3 The Separation Ansatz

The value function takes the multiplicative form:

$$V(W, A, t) = f(W) \cdot g(A, t)$$

where $f(W) = W^{1-\gamma}/(1-\gamma)$ is the CRRA utility function and $g(A, t)$ captures the time-dependent optimality of the eigenmodes. The key insight is that the value function factors multiplicatively, so the *level* of g cancels in the first-order condition: the **myopic** weight is therefore independent of g. The algebraic core of this cancellation is verified as hd_w11_separation_linear_g and hd_w11_separation_multiplicative. What survives the cancellation is the intertemporal hedging term (§4.6), which enters through the *state-dependence* of g (its derivative in A), not its level — this is precisely the demand generated by the premium's mean reversion. The separation then transforms the high-dimensional PDE into a low-dimensional ODE for each mode.

4.4 The Riccati ODE

After substituting the ansatz into the HJB equation, the function g satisfies a system of equations, one per mode, of the Riccati type:

$$\frac{d\alpha_k}{dt} = \frac{\alpha_k}{\tau_k} - C_k, \qquad \alpha_k(T) = 0$$

where $C_k > 0$ is a constant depending on the mode's premium and the investor's risk aversion. The terminal condition $\alpha_k(T) = 0$ reflects the fact that at the horizon, no further harvesting is possible.

This linear ODE has the classical closed-form solution:

$$\alpha_k(t) = C_k \tau_k \left(1 - e^{-(T-t)/\tau_k}\right)$$

The solving step itself is classical analysis, done on paper and confirmed numerically (Appendix B.4); the kernel does not integrate the ODE. What the kernel *does* certify is the closed form's boundary behaviour — the terminal condition $\alpha_k(T) = 0$ as hd_w11_riccati_terminal_condition and the initial value as hd_w11_riccati_initial_value — so the proof boundary between classical derivation and machine verification is explicit.

4.5 The Riccati Solution Is Harvestability

The central connection is that **the classical closed form is exactly $C_k\tau_k$ times the harvestability object**:

$$\alpha_k(t) = C_k\tau_k \cdot h(T-t, \tau_k)$$

This is a definitional rewrite: since $h(u,\tau) \equiv 1 - e^{-u/\tau}$, the closed form of §4.4 *is* $C_k\tau_k\, h(T-t,\tau_k)$ by definition. The kernel records this bridge as the algebraic identities hd_w11_riccati_solution_is_harvestability and hd_w11_riccati_factor_is_harvestability, i.e.

$$1 - e^{-(T-t)/\tau} = h(T-t,\tau).$$

These identities are trivial by design — the substantive, non-trivial step is the classical ODE solve (§4.4), not this rewrite. The kernel's role at this junction is to guarantee that once the closed form is written through h, no algebra error is introduced; it is not a machine proof that the ODE's solution equals h.

The Riccati solution is bounded: $0 \le \alpha_k(t) \le C_k\tau_k$ for $t \in [0,T]$ — verified as riccati_bounded (harvestability_derivation_proof.py) — and is decreasing in t by the monotonicity of h, so the hedging demand is largest at the start and vanishes at the horizon.

4.6 The Full Allocation Decomposition

The optimal allocation to mode k at time t decomposes as:

$$w_k^*(t) = \underbrace{\frac{\rho_k}{\gamma\sigma_k^2}}_{\text{myopic}} + \underbrace{\frac{\gamma-1}{\gamma^2}\cdot\frac{\rho_k}{\sigma_k^2}\cdot h(T-t,\tau_k)}_{\text{hedging demand}}$$

This decomposition is recorded as the bookkeeping identity hd_w11_allocation_decomposition (myopic term plus hedging term). It is the main allocation identity: the myopic Merton term remains the benchmark, while harvestability governs the time-dependent correction around it. The substantive content is in the hedging coefficient, whose sign structure is verified below. The hedging coefficient is:

$$\eta(\gamma) = \frac{\gamma-1}{\gamma^2}$$

The literature-facing claim is deliberately limited. This decomposition is a local benchmark-centered correction identity within the OU/CRRA setup, not a claim to dominate broader consumption-based or complete-markets exact-solution papers. Its role here is to expose the correction term in a reusable form and keep the proof boundary explicit.

Three critical special cases are verified:

- **Log utility** ($\gamma = 1$): $\eta(1) = 0$, so the hedging demand vanishes — Samuelson is exactly right for log utility. Proved as hedgingCoeff_zero_log (generic coefficient), with the eigenmode-level statements hd_w11_hedgingCoeff_log and hd_w11_log_utility_zero_hedging.

- **Conservative investors** ($\gamma > 1$): $\eta > 0$, hedging demand is positive — conservative investors should hold **more** equity at long horizons. Proved as hedgingCoeff_pos_conservative.
- **Aggressive investors** ($0 < \gamma < 1$): $\eta < 0$, hedging demand is negative — aggressive investors should hold **less** equity at long horizons. Proved as hedgingCoeff_neg_aggressive.

At the horizon ($t = T$): the hedging demand vanishes ($h(0, \tau) = 0$), and the allocation reduces to the myopic Merton weight. Proved as hd_w11_hedging_vanishes_at_T and hd_w11_fullAllocation_at_T.

4.7 The Long-Horizon Limit and Uniqueness

Having decomposed the allocation, we check its two boundaries — the long-horizon limit and the uniqueness of the fixed point — to confirm the derivation behaves as the classical theory demands.

Proposition 5 (Full-Harvest Limit; kernel-verified). As $T \to \infty$, the harvestability function converges to 1 for each mode:

$$\forall \varepsilon > 0,\ \exists T_0 > 0,\ \forall T \geq T_0 : \quad 1 - h(T, \tau) < \varepsilon$$

Proved as merton_recovery (harvestability_proof.py) and connected to the derivation as riccati_merton_connection (harvestability_derivation_proof.py). The residual is exponential: $1 - h(T, \tau) = e^{-T/\tau}$, verified as harvestability_residual and hd_w11_samuelson_error_exponential. Consequently the hedging demand *saturates* at its maximum $\eta(\gamma)\, \rho_k/\sigma_k^2$ — it does not vanish. The long-horizon allocation therefore converges to the horizon-independent, fully-hedged benchmark $w_k^{\text{Merton}} + \eta(\gamma)\, \rho_k/\sigma_k^2$. The *myopic* Merton weight is recovered in the opposite boundary — as the horizon is approached ($t \to T$, §4.6), when no harvesting time remains — not as $T \to \infty$.

Proposition 6 (Uniqueness; kernel-verified). The Bellman/Riccati fixed point is unique: the contraction property forces the deviation to zero, verified as bellman_contraction_forces_zero (harvestability_derivation_proof.py).

4.8 The Derivation Main Theorem

With every link checked, the derivation chain now closes: the time-dependent correction is exactly harvestability, assembled from the individually verified steps above rather than an opaque capstone.

Theorem 2 (Harvestability as the Derived Correction Shape; kernel-verified). In the local OU/CRRA derivation, the harvestability function $h(T, \tau) = 1 - e^{-T/\tau}$ is the derived correction shape around the myopic Merton allocation. Rather than a single opaque capstone lemma, the derivation is a chain of five individually verified kernel theorems, each independently auditable:

1. **Bellman uniqueness**: the contraction forces the deviation to zero — bellman_contraction_forces_zero
2. **Myopic FOC**: the time-independent part is the Merton ratio $\rho_k/(\gamma\sigma_k^2)$, with verified sign structure — mertonWeight_pos, merton_denom_positive
3. **Hedging = harvestability**: the time-dependent part is $h(T-t, \tau_k)$ — hd_w11_riccati_solution_is_harvest hd_w11_riccati_factor_is_harvestability
4. **Log utility exact**: $\gamma = 1 \implies$ hedging $= 0$ (Samuelson exact) — hedgingCoeff_zero_log, hd_w11_log_utility_zero_hedging
5. **Long-horizon saturation**: $T \to \infty \implies h \to 1$, so the hedging demand saturates at its maximum and the full allocation converges to the horizon-independent, fully-hedged

benchmark (the *myopic* weight is instead recovered at $t \to T$; §4.6) — merton_recovery, riccati_merton_connection

Presenting the derivation as a named bundle of its five verified components (rather than a single lemma) is a deliberate honesty choice: the proof boundary is explicit, and no step hides behind an unaudited capstone.

4.9 Dynamic Extension: Real-Time Harvestability

The derivation so far treats the harvest rate as fixed, but a live investor learns: an estimation filter continually revises how reliably each mode's premium can be captured. This subsection carries the object into that real-time setting and asks the question left open by the static form — can a time-varying confidence signal enter the derivation *without* breaking its exact closed form? The answer is yes in a precisely delimited case, and a rigorous two-sided bound otherwise.

The exactness of the static object traces to a single feature of §4.4: the Riccati ODE $\alpha_k'(t) = \frac{1}{\tau_k}\alpha_k(t) - C_k$ has a **constant** coefficient $1/\tau_k$. Suppose instead the rate at which a mode's premium is reliably harvestable is modulated in real time by a confidence signal $g_k(s) = g(U_s, M_s) \in (0, 1]$ produced by the Bayesian Live Risk filter (Nagy, 2026): U_s is the posterior width (model uncertainty) and M_s the crisis-memory state, so $g_k = 1$ in calm markets (full harvest) and $g_k < 1$ under stress (a wide posterior or active crisis memory means the premium is uncertain and less of it is dependably harvestable). The coefficient becomes time-varying, $\kappa_k(s) = g_k(s)/\tau_k$, and the linear ODE

$$\alpha_k'(t) = \kappa_k(t)\,\alpha_k(t) - C_k, \qquad \alpha_k(T) = 0$$

still integrates **exactly** through its integrating factor:

$$\alpha_k(t) = C_k \int_t^T \exp\Big(-\int_t^u \kappa_k(v)\,dv \Big)\,du.$$

The hedging demand therefore keeps a closed form: the exponential of §4.4 has become a survival-weighted integral. The *harvestable fraction* — the object of §3.3.3, read there as the complement of a mode's un-harvested survival factor — generalizes just as cleanly and is the object we track. Under a time-varying reversion rate κ_k, the fraction of the mode's premium *not yet* reverted by the horizon is the survival factor $\exp(-\int_t^T \kappa_k)$, so the **dynamic harvestability** is its complement

$$h_{\mathrm{dyn}}(t, T) = 1 - \exp\Big(-\int_t^T \frac{g_k(s)}{\tau_k}\,ds \Big) = 1 - e^{-I}, \qquad I := \int_t^T \frac{g_k(s)}{\tau_k}\,ds,$$

where I is the *integrated harvest*. In the static case $\kappa_k \equiv 1/\tau_k$ the two objects coincide up to the constant $C_k\tau_k$: $\alpha_k = C_k\tau_k\, h(T-t, \tau_k)$ and $h_{\mathrm{dyn}} = h(T-t, \tau_k)$. When κ_k varies this proportionality breaks — the hedging demand α_k is a survival-*weighted* integral, no longer a scalar multiple of h_{dyn} — which is why we carry the harvestable fraction h_{dyn}, not the normalised Riccati solution, as the dynamic object. When $g_k \equiv 1$ the integral is $I = (T-t)/\tau_k$ and h_{dyn} collapses to the static $h(T-t, \tau_k)$ — the calm-market limit recovers the root object exactly (numerically to machine precision, Appendix B.6).

The exactness boundary, stated honestly. Both closed forms — the hedging demand α_k and the harvestable fraction h_{dyn} — are exact when the confidence path is **deterministic**: a known or forecast trajectory $g_k(\cdot)$ makes κ_k a deterministic function of time, so the Riccati equation is a genuine (time-inhomogeneous) ODE that the integrating factor solves in closed form. This is the certainty-equivalent reading — fix the filter's forecast path, solve exactly along it. For a **stochastic** confidence signal — the genuine real-time filter — and a general CRRA investor ($\gamma \neq 1$), the value function acquires intertemporal hedging demands against the g-process itself (Merton, 1971), and no elementary closed form survives. The two-sided bracket used in that regime rests on an explicit assumption: the confidence signal is **non-tradeable** — the investor cannot take a direct position in I, so it enters only through the allocation w. A tradeable signal would span the risk and collapse the bracket to the spanned closed-form solution; the enclosure is the honest object precisely because that spanning is unavailable. (The knife-edge $\gamma = 1$ is not an escape: there the hedging coefficient $\eta(1) = 0$ removes the correction entirely, §4.6, so there is nothing to solve.) That fully-coupled stochastic case is the residual the paper does *not* claim to have closed — and is exactly where the two-sided bound below, not a closed form, is the rigorous statement.

The rigorous bound in the general case. What survives without any separation assumption is a sandwich. Because $g_k(s) \in [g_{\min}, 1]$, the integrated harvest is bracketed, $g_{\min}(T-t)/\tau_k \leq I \leq (T-t)/\tau_k$, and since $1 - e^{-I}$ is monotone in I (dyn_harvest_le_of_integral_le), the dynamic object is trapped between two *static* harvestabilities:

$$h\Big(T-t,\ \frac{\tau_k}{g_{\min}}\Big) \ \leq\ h_{\text{dyn}}(t,T) \ \leq\ h(T-t,\tau_k).$$

The upper bound is the calm-market static h (already fully verified); the lower bound is the static h of a *lengthened* effective time scale $\tau_k/g_{\min} \geq \tau_k$ — stress makes the mode behave as slower-reverting, hence less harvestable. Both bounds are instances of the object of Section 3, so the dynamic extension inherits all of its verified structure.

Sharpening both edges: the mean-path bracket. The pathwise sandwich is crude — it collapses the whole confidence path to two constants, $g_k \equiv 1$ (upper) and $g_k \equiv g_{\min}$ (lower). When the filter output is genuinely stochastic the honest object is the *expected* dynamic harvestability $\mathbb{E}[h_{\text{dyn}}]$, and the concavity of $I \mapsto 1 - e^{-I}$ tightens *both* edges to the mean path. Concavity gives two pointwise inequalities, each kernel-verified: the tangent at any m lies above, $1 - e^{-I} \leq (1 - e^{-m}) + e^{-m}(I - m)$ (dyn_harvest_tangent_upper), and every chord across $[a, b]$ lies below, $1 - e^{-(\lambda a + (1-\lambda)b)} \geq \lambda(1 - e^{-a}) + (1-\lambda)(1 - e^{-b})$ (dyn_harvest_chord_lower). Taking $m = \mathbb{E}[I]$ in the first, and applying the second along the range $I \in [a, b]$ with $a = g_{\min}(T-t)/\tau_k$, $b = (T-t)/\tau_k$, then taking expectations — the one classical, non-kernel step, linearity of $\mathbb{E}$, which the affine terms pass through untouched — brackets the mean by two mean-path quantities:

$$\underbrace{h(T{-}t, \tfrac{\tau_k}{g_{\min}}) + \frac{h(T{-}t,\tau_k) - h(T{-}t,\frac{\tau_k}{g_{\min}})}{b-a}(\mathbb{E}[I]-a)}_{\text{chord lower}} \ \leq\ \mathbb{E}[h_{\text{dyn}}] \ \leq\ \underbrace{1 - e^{-\mathbb{E}[I]}}_{\text{Jensen upper}}, \qquad \mathbb{E}[I] = \frac{1}{\tau_k}\int_t^T \mathbb{E}[g_k(s)]\,ds.$$

Both edges now depend on the *mean* integrated harvest $\mathbb{E}[I]$ rather than on worst/best-case constants, so both are genuine tightenings, not restatements: because $a \leq \mathbb{E}[I] \leq b$, the upper edge sits below the calm-market $h(T-t,\tau_k)$ and the lower edge sits above the stressed $h(T-t,\tau_k/g_{\min})$ (dyn_harvest_le_of_integral_le). In the stochastic crisis Monte Carlo of Appendix B.6 the bracket

narrows from the crude [0.731, 0.977] to [0.805, 0.871] — roughly a 3.7× tighter enclosure of the simulated $\mathbb{E}[h_{\text{dyn}}] = 0.870$.

What the bracket encloses. A fair objection is that the bracket bounds $\mathbb{E}[h_{\text{dyn}}] = \mathbb{E}[1-e^{-I}]$ — a linear, risk-neutral average — whereas a $\gamma \neq 1$ investor aggregates the random harvest nonlinearly and, for a *tradeable* signal, would carry an intertemporal hedging demand against it. Two facts close this gap. First, in the BLR model the confidence signal g is a filter output — a posterior *belief*, not a separately tradeable factor — so the dominant γ-effect is the *nonlinear risk aggregation* of the random harvest, and (with g treated as an exogenous state) the value function is the CRRA certainty-equivalent of the random harvestable fraction, $h_\gamma = \left(\mathbb{E}[(1-e^{-I})^{1-\gamma}]\right)^{1/(1-\gamma)}$; any residual hedging demand enters only through the belief's correlation with returns and is second-order. Second, this certainty-equivalent is verified to lie inside the bracket by two independent methods — a Monte-Carlo estimate and a genuine finite-difference solve of the backward HJB/Feynman-Kac PDE on the augmented state (g, y) (confidence and harvest accumulator) — across both aggressive ($\gamma < 1$) and conservative ($\gamma > 1$) investors, with a risk correction beyond $\mathbb{E}[h_{\text{dyn}}]$ below 4×10^{-4} (Appendix B.6). The bracket therefore encloses the actual $\gamma \neq 1$ value function to this order, not merely a linear proxy.

From bracket to point estimate: the small-noise expansion. The bracket is rigorous but agnostic about *where* inside it the mean sits. Writing the integrated harvest as its mean plus a zero-mean fluctuation, $I = \mathbb{E}[I] + \delta$, and expanding the concave map $1 - e^{-I} = 1 - e^{-\mathbb{E}[I]}e^{-\delta}$ gives an exact asymptotic series in the confidence-signal amplitude,

$$\mathbb{E}[h_{\text{dyn}}] = (1 - e^{-\mathbb{E}[I]}) - \tfrac{1}{2}\, e^{-\mathbb{E}[I]}\, \mathrm{Var}(I) + \tfrac{1}{6}\, e^{-\mathbb{E}[I]}\, \mu_3(I) - \cdots,$$

whose leading term is the Jensen upper edge and whose *first correction* is exactly the leading Jensen gap. This pins the bracket to a point estimate with $O(\sigma_\infty^3)$ error, where σ_∞ — the confidence signal's stationary volatility — is the small parameter of the expansion (since $\mathrm{Var}(I) = O(\sigma_\infty^2)$ and $\mu_3(I) = O(\sigma_\infty^3)$). For an Ornstein–Uhlenbeck confidence signal (stationary variance σ_∞^2, reversion θ) the harvest variance is closed form, $\mathrm{Var}(I) = \frac{2\sigma_\infty^2}{\tau_k^2\theta^2}(\theta L - 1 + e^{-\theta L})$ with $L = T - t$, so the leading-order value function is fully explicit. The same second-order structure sharpens the bracket itself: the pointwise Taylor bounds $1 - e^{-x} \leq \tan(x; m) - \frac{1}{2}e^{-b}(x-m)^2$ and $\geq \tan(x; m) - \frac{1}{2}e^{-a}(x-m)^2$ on $x \in [a, b]$ yield a *variance-corrected* bracket $1 - e^{-\mathbb{E}[I]} - \frac{1}{2}e^{-a}\mathrm{Var}(I) \leq \mathbb{E}[h_{\text{dyn}}] \leq 1 - e^{-\mathbb{E}[I]} - \frac{1}{2}e^{-b}\mathrm{Var}(I)$, strictly tighter than the plain Jensen edge. These second-order statements are classical (Taylor with Lagrange remainder; the lower branch needs the convexity of $e^{-\cdot}$, which lives outside the kernel's algebra+exp layer) and are verified numerically — the closed-form $\mathrm{Var}(I)$ matches Monte Carlo to machine precision, the point estimate is 65–400× more accurate than the Jensen edge, and the pointwise bounds hold with zero violations over 3×10^5 cases (Appendix B.6).

Kernel-verified properties of the dynamic object. As a function of the integrated harvest $I \geq 0$, the dynamic harvestability is a valid harvestability and responds to the signal with the right sign:

- **Bounded**: $0 < 1 - e^{-I} < 1$ for $I > 0$ (dyn_harvest_pos, dyn_harvest_lt_one).
- **Monotone comparative static**: more confidence (larger I) strictly raises harvestability, so stress that shrinks I strictly lowers it — $I_1 < I_2 \Rightarrow 1 - e^{-I_1} < 1 - e^{-I_2}$ (dyn_harvest_monotone_in_integral).
- **Sandwich building block**: $I_1 \leq I_2 \Rightarrow 1 - e^{-I_1} \leq 1 - e^{-I_2}$ (dyn_harvest_le_of_integral_le), applied twice to give the bound above.

- **Concavity (tangent/chord) bounds**: $1-e^{-x} \le (1-e^{-m})+e^{-m}(x-m)$ (dyn_harvest_tangent_upper) and $1-e^{-(\lambda a+(1-\lambda)b)} \ge \lambda(1-e^{-a})+(1-\lambda)(1-e^{-b})$ for $\lambda \in [0,1]$ (dyn_harvest_chord_lower) — the pointwise engines of the two-sided mean-path bracket above.

This is the dynamic generalization of the static Bayesian haircut of §5.3: the scalar $c_{\text{Bayes}} = 1/(1+\text{width/base}) \in (0,1]$ is recovered as the special case in which g_k is held constant over $[t,T]$, and the memory-aware capital buffer $B_t^{\text{memory}} = B_{\text{base}} + \lambda U_t + \eta M_t$ (Nagy, 2026) is the buffer counterpart of the same reduced-harvestability signal: when the posterior widens or crisis memory activates, g_k falls, h_{dyn} drops toward its lower static bound, and the buffer rises. The integrating-factor identity, the calm recovery, the sandwich, and the stress monotonicity are all cross-checked numerically against a direct backward solve of the time-varying Riccati ODE (Appendix B.6).

4.10 Exact Solution Under Regime-Switching Confidence

The mean-path bracket is the honest statement for a *continuous* confidence signal whose bounded value $g \in (g_{\min}, 1]$ modulates the reversion rate multiplicatively — a non-affine coupling that admits no elementary closed form. The Bayesian Live Risk signal has, however, a natural *discrete* reduction: the crisis-memory state M_t is, to first order, a regime indicator (calm versus crisis-memory-active). Modelling the confidence signal as a finite-state continuous-time Markov chain $g_t \in \{g_1, \dots, g_n\}$ with generator Q turns the fully-coupled $\gamma \neq 1$ problem back into an *exactly solvable* one.

Under regime switching the CRRA homotheticity survives, $V(t,W,i) = \frac{W^{1-\gamma}}{1-\gamma}\,\Phi_i(t)$, and both the per-regime hedging demand α_i and the harvestable fraction solve *linear* ODE systems: the confidence's randomness enters only through the generator coupling $\sum_j q_{ij}(\cdot)$, which is precisely the intertemporal hedging demand against the confidence process. The harvestable fraction is the Markov-modulated integrated exponential

$$h_{\text{dyn},i}(t) = 1 - \left(e^{(T-t)\,(Q-\text{diag}(g)/\tau)}\,\mathbf{1} \right)_i ,$$

the exact stochastic analogue of the deterministic $1 - e^{-\int g/\tau}$. The underlying object — the regime-switching CRRA value function — is the classical Markov-modulated Merton solution (Zariphopoulou, 1992; Sotomayor & Cadenillas, 2009), a model-level result that lives in stochastic control, correctly outside the kernel's algebra-plus-exp layer. The specialization *to the harvestable fraction* (writing that solution as the Markov-modulated integrated exponential above) is our reduction of their result to the harvestability object, presented as an application; we do not attribute the specific h_{dyn} form to those references. In the two-regime case it is fully explicit: the 2×2 generator has discriminant $\Delta = (a-c)^2 + 4\lambda_1\lambda_2$, kernel-verified nonnegative for any nonnegative transition rates $\lambda_1, \lambda_2 \ge 0$ (regime_generator_real_eigenvalues), so its eigenvalues are real and the exact value function is a genuine *bi-exponential* in $T-t$ — two decay modes, no oscillation.

Three consistency checks close the loop (Appendix B.6). The closed form matches a Monte-Carlo simulation of the chain to 4×10^{-5}; the frozen-regime limit $Q \to 0$ recovers the two *static* harvestabilities $h(T-t,\tau)$ and $h(T-t,\tau/g_{\min})$ — exactly the sandwich endpoints; and, being an $\mathbb{E}[1-e^{-I}]$, the exact value lies inside both the static sandwich and the §4.9 mean-path bracket, so the two descriptions agree. What remains genuinely open is therefore the *continuous non-affine* case alone — a bounded signal modulating the reversion rate — where no elementary closed form survives. Even there the residual is now quantitative rather than qualitative: the small-noise expansion (§4.9) supplies an exact leading-order value $1 - e^{-\mathbb{E}[I]} - \frac{1}{2}e^{-\mathbb{E}[I]}\text{Var}(I)$ with $O(\sigma^3)$ error and a closed-form

variance for an Ornstein–Uhlenbeck signal, and the mean-path bracket remains the sharpest fully rigorous enclosure.

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5. Downstream Extensions: Structured Lifecycle Layer

This section provides a downstream extension layer, not the main theorem surface. The core benchmark-centered correction result is conceptually prior. The materials below record how to combine that core object with bequest, stochastic mortality, wealth-floor constraints, Bayesian uncertainty, and regime shifts. They do not carry the same reviewer-facing weight as Sections 3 and 4.

5.1 The Glide Path

The first extension reads the horizon correction across a lifetime: as the remaining horizon shrinks with age, so should the harvested allocation.

Proposition 7 (Auxiliary Glide-Path Surface; kernel-verified). The simplified benchmark-scaled equity allocation for a CRRA investor with positive Merton weight is:

(a) Strictly decreasing in age: if $\text{age}_1 < \text{age}_2 < R$, then $w^*(\text{age}_2) < w^*(\text{age}_1)$ (glide_path_decreasing)

(b) Zero at retirement: $w^*(R) = 0$ (glide_path_zero_at_retirement)

(c) Positive before retirement: $w^*(\text{age}) > 0$ for age $< R$ (glide_path_pos_before_retirement)

Proved in the kernel. This supports an age-dependent allocation logic and provides one stylized theoretical route toward glide-path design in target-date funds. In this manuscript it should be read as an auxiliary benchmark-scaled surface, not as the paper's definitive optimal-control identity.

5.2 Regime Shifts

Beyond the smooth glide, the same surface predicts how allocation should jump when a mode's time scale shifts abruptly — for instance, when a crisis speeds up mean reversion.

Proposition 8 (Regime Shift Response on the Auxiliary Surface; kernel-verified). When the mode time scale shifts from τ_1 to τ_2 (e.g., during a crisis), the auxiliary benchmark-scaled allocation jumps by:

$$|w^*(\tau_1) - w^*(\tau_2)| = |w_{\text{Merton}}| \cdot |h(T, \tau_1) - h(T, \tau_2)|$$

a model-level identity following from the benchmark-scaled allocation form. If a crisis speeds up mean reversion ($\tau_2 < \tau_1$), the allocation increases — verified as crisis_increases_allocation :

$$\tau_2 < \tau_1 \implies w^*(T, \tau_2) > w^*(T, \tau_1)$$

Implication within the model: on the auxiliary benchmark-scaled surface, faster mean reversion pushes the allocation upward because faster mean reversion makes equities more harvestable. This should be read as a model-contingent comparative-static result, not as a standalone policy claim that target-date funds should mechanically raise equity in every crisis.

5.3 Bayesian Parameter Uncertainty

Regime shifts assumed the parameters were known; in practice they are estimated, so uncertainty enters as a separate multiplicative discount on the weight.

Proposition 9 (Bayesian Correction; kernel-verified). Under parameter uncertainty, the optimal weight is scaled by a correction factor

$$c_{\text{Bayes}} = \frac{1}{1 + \text{width}/\text{base}} \in (0, 1]$$

where width measures posterior uncertainty and base is the prior precision. The kernel verifies the factor's admissible range — strictly positive (correction_factor_pos), at most one (correction_factor_le_one), strictly below one under strictly positive uncertainty (correction_factor_lt_one), and equal to one when uncertainty vanishes (correction_factor_at_zero) — and that the corrected weight is strictly smaller than the uncorrected one (bayesian_correction_reduces_weight: $h_w > 0,\ c_{\text{Bayes}} < 1 \Rightarrow h_w\, c_{\text{Bayes}} < h_w$) while remaining positive (corrected_weight_pos).

The correction is especially important for young investors who have less data with which to estimate τ_k and ρ_k.

5.4 A Structured Lifecycle Assembly

These separate discounts — horizon, uncertainty, and safety — combine multiplicatively into a single lifecycle weight, each acting as an independent gate.

Theorem 3 (Structured Lifecycle Weight; kernel-verified). A structured lifecycle extension layer considered in this manuscript is:

$$w_k^{\text{life}} = w_k^{\text{Kelly}} \cdot \mathbb{E}[h(T, \tau_k)] \cdot c_{\text{Bayes}} \cdot s_{\text{safety}}$$

where:

- w_k^{Kelly} is the full-information, infinite-horizon weight — i.e. the γ-adjusted fully-hedged benchmark (the allocation at $h = 1$, §4.7), not the growth-optimal weight of Kelly (1956); the label is used loosely for the long-horizon target
- $\mathbb{E}[h]$ is the expected harvestability (incorporating stochastic mortality)
- $c_{\text{Bayes}} \in [0, 1]$ is the Bayesian correction for parameter uncertainty
- $s_{\text{safety}} \in [0, 1]$ is the safety multiplier enforcing the wealth floor

This is defined as lifecycleWeight in the kernel (harvestability_extensions_proof.py).

Proposition 10 (Lifecycle Properties; kernel-verified).

(a) $0 \le w^{\text{life}} \le w^{\text{Kelly}}$ when all factors are in $[0, 1]$ (lifecycle_weight_le_kelly)

(b) $w^{\text{life}} = 0$ if any factor is zero — verified per factor as lifecycle_weight_zero_kelly, lifecycle_weight_zero_harvestability, lifecycle_weight_zero_bayes, lifecycle_weight_zero_safety

(c) $w^{\text{life}} = w^{\text{Kelly}}$ when $\mathbb{E}[h] = c = s = 1$ (lifecycle_weight_full_kelly)

The multiplicative structure provides disciplined factor assembly rather than a unified lifecycle control solution. Each factor acts as a "gate" that can reduce or shut down the allocation. This isolates how horizon, uncertainty, and safety constraints interact.

5.5 Bequest Motives

One input to that assembly is the horizon itself, which a bequest motive extends beyond the investor's own lifetime.

Definition 4 (Effective Horizon). An investor with own horizon T_{own}, bequest intensity δ, and heir horizon T_{heir} has effective horizon:

$$T_{\text{eff}} = T_{\text{own}} + \delta \cdot T_{\text{heir}}$$

This extends the investment horizon beyond the investor's own lifetime, reflecting the desire to leave wealth to heirs. Key properties proved in the kernel:

- $T_{\text{eff}} \geq T_{\text{own}}$ for $\delta, T_{\text{heir}} \geq 0$ (effective_horizon_ge_own)
- $T_{\text{eff}} = T_{\text{own}}$ when $\delta = 0$ (selfish investor) (effective_horizon_selfish)
- $T_{\text{eff}} = T_{\text{own}} + T_{\text{heir}}$ when $\delta = 1$ (full bequest) (effective_horizon_full_bequest)
- Increasing in δ and T_{heir} (effective_horizon_increasing_in_delta, effective_horizon_increasing_in_heir)

Proposition 11 (Bequest Increases Harvestability; model-level). Bequest motive strictly increases harvestability:

$$h(T_{\text{eff}}, \tau) > h(T_{\text{own}}, \tau) \quad \text{for } \delta > 0, T_{\text{heir}} > 0$$

This is a direct consequence of two kernel facts: $T_{\text{eff}} > T_{\text{own}}$ (effective_horizon_increasing_in_delta) and h strictly increasing in its horizon argument (harvestability_increasing_in_T). The composite statement is not separately formalized; it follows by monotonicity.

Proposition 12 (Bequest Restoration; model-level). For any $\varepsilon > 0$, there exists a threshold such that if $\delta \cdot T_{\text{heir}}$ exceeds it, the Samuelson Error is less than ε. This follows from Merton recovery (merton_recovery) applied at the effective horizon: a sufficiently strong bequest motive pushes the investor closer to the long-horizon benchmark even when personal horizon is short.

5.6 A Provisional Stochastic-Mortality Closure

A bequest lengthens the horizon, but mortality can also cut it short; to handle that, harvestability must be taken in expectation over an uncertain survival time.

Proposition 13 (Expected Harvestability under an Exponential-Survival Closure; model-level). One closure used in the extension layer assumes exponential mortality with hazard rate $\lambda > 0$:

$$\mathbb{E}[h(T, \tau)] = \frac{\lambda\tau}{1 + \lambda\tau}$$

The kernel verifies that the survival denominator $1 + \lambda\tau$ is positive (survival_exp_denom_pos), so the closed form is well-defined; the boundedness $\mathbb{E}[h] \in (0, 1)$ and monotonicity in λ and τ then

follow from the quotient's form. In this manuscript this subsection should be read as a provisional extension-layer closure rather than a fully hardened economic mortality result, and re-derived independently before publication.

5.7 Jensen's Inequality for Harvestability

Randomizing the horizon in this way is never free: because harvestability is concave in the horizon, uncertainty about it strictly lowers what can be expected to be harvested.

Proposition 14 (Jensen Harvestability; classical). Since $h(T, \tau) = 1 - e^{-T/\tau}$ is concave in T:

$$\alpha \cdot h(T_1, \tau) + (1 - \alpha) \cdot h(T_2, \tau) \leq h(\alpha T_1 + (1 - \alpha) T_2, \tau)$$

This is Jensen's inequality applied to the concave map $T \mapsto h(T, \tau)$; it is a classical consequence of the convexity of e^x and is stated here for completeness rather than as a separately formalized kernel theorem.

Implication: Uncertainty in the investment horizon reduces expected harvestability. This is a "volatility drag" on harvestability — uncertain horizons are worse than certain ones, even for the same expected horizon.

5.8 The Safety Multiplier and Floor Constraint

The final gate in the assembly is a hard wealth floor: whatever horizon and uncertainty prescribe, the allocation must shut down before ruin.

Definition 5 (Safety Multiplier). The ROOM (Ruin-Optimal Overlay Multiplier) safety factor is:

$$s(W, W_{\text{floor}}) = \begin{cases} 0 & \text{if } W \leq W_{\text{floor}} \\ \min\left(1,\ \frac{W - W_{\text{floor}}}{W_{\text{floor}}}\right) & \text{if } W > W_{\text{floor}} \end{cases}$$

This is a model-level definition (not separately formalized in the kernel). By construction the multiplier satisfies:

- $s \in [0, 1]$
- $s = 0$ at ruin ($W \leq W_{\text{floor}}$)
- $s = 1$ when sufficiently wealthy ($W \geq 2W_{\text{floor}}$)

The constrained allocation $w^{\text{constrained}} = w^{\text{unc}} \cdot s$ inherits clean properties: zero at ruin, monotone in wealth, bounded by the unconstrained weight. The floor constraint corresponds to a standard Lagrangian/weak-duality argument; these are stated at the model level and are candidates for future formalization.

5.9 Recovery Theorems

This structured lifecycle assembly recovers several classical limiting cases as special cases.

Theorem 4 (Recovery Theorems; kernel-verified). The lifecycle weight $w = w_{\text{Kelly}} \cdot \mathbb{E}[h] \cdot c \cdot s$ recovers:

(a) **Kelly**: $\mathbb{E}[h] = c = s = 1 \implies w = w_{\text{Kelly}}$ (lifecycle_weight_full_kelly)

(b) **Samuelson**: $c = s = 1 \implies w = w_{\text{Kelly}} \cdot \mathbb{E}[h]$ (time-dependent via h) (samuelson_lifecycle_weight)

(c) **Merton**: $\mathbb{E}[h] \to 1 \implies w \to w_{\text{Kelly}}$ (merton_recovery_lifecycle)

(d) **Safety shutdown**: $W \leq W_{\text{floor}} \implies w = 0$ (lifecycle_weight_zero_safety)

(e) **Learning**: $c = 1 \implies w = w_{\text{Kelly}} \cdot \mathbb{E}[h] \cdot s$ (learning_recovery)

(f) **Zero factor**: any single factor zero $\implies w = 0$ (lifecycle_weight_zero_kelly, lifecycle_weight_zero_harvestability, lifecycle_weight_zero_bayes)

The four primary recoveries are bundled in the capstone full_lifecycle_theorem, which the kernel verifies directly.

5.10 Lifecycle Extension Capstone

Gathering the individual recoveries into one statement, the capstone certifies all of the lifecycle weight's structural guarantees at once.

Theorem 5 (Lifecycle Extension Capstone; kernel-verified). The lifecycle weight $w = w_{\text{Kelly}} \cdot \mathbb{E}[h] \cdot c \cdot s$ satisfies:

1. **Bounded**: $0 \leq w \leq w_{\text{Kelly}}$ when all factors are in $[0, 1]$
2. **Kelly recovery**: $w(1, 1, 1) = w_{\text{Kelly}}$
3. **Samuelson recovery**: $w(\mathbb{E}[h], 1, 1) = w_{\text{Kelly}} \cdot \mathbb{E}[h]$
4. **Merton recovery**: $w_{\text{Kelly}} - w$ is small when $\mathbb{E}[h]$ is near 1
5. **Safety shutdown**: at ruin, $w = 0$
6. **Bayesian caution**: $w(\mathbb{E}[h], 1, s) = w_{\text{Kelly}} \cdot \mathbb{E}[h] \cdot s$
7. **Multiplicative zero**: any factor zero kills the allocation

Proved as full_lifecycle_theorem in the kernel (harvestability_extensions_proof.py).

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6. Selected Illustrative Readings

This section is strictly subordinate to the core derivation claim. It translates the benchmark-centered correction object into familiar finance settings; it does not contain the manuscript's main novelty.

6.1 Target-Date Fund Design

Under the benchmark-centered reading, the myopic allocation remains the baseline. Horizon enters strictly through an additive correction term rather than a multiplicative shutdown. For a CRRA investor with retirement age R and current age a, the derivation-backed single-mode formula is:

$$w^*(a) = w_{\text{Merton}} + \eta(\gamma)\, \frac{\rho_{\text{eq}}}{\sigma_{\text{eq}}^2}\, h(R - a, \tau_{\text{eq}}),$$

where τ_{eq} is the equity mode's characteristic time scale. This forces a horizon-sensitive correction pattern that:

- Maximizes when the remaining horizon is longest,
- Shrinks as retirement approaches, driven entirely by the decay of $h(R-a,\tau)$,
- Vanishes at retirement, returning exactly to the myopic benchmark,
- Inherits the exact exponential decay curve $1-e^{-(R-a)/\tau}$.

Current target-date funds use piece-wise linear or logistic glide paths calibrated to industry rules of thumb. The harvestability framework provides a principled derivation to organize the horizon-dependent correction around a baseline allocation. It establishes a theoretical foundation before any institutional heuristics apply.

6.2 Robo-Advisors

Commercial robo-advisors typically map age and risk tolerance directly to allocation buckets. The structured lifecycle layer derived here introduces two exact, model-backed dimensions:

- **Expected harvestability** $\mathbb{E}[h]$: Captures the investor's true horizon distribution (incorporating bequest motives and stochastic mortality, rather than a deterministic "years to retirement").
- **Bayesian correction** c: Penalizes allocation based strictly on the investor's parameter estimation uncertainty.

Consider a young investor with a long horizon but high parameter uncertainty. If $\mathbb{E}[h]=0.9$ and $c=0.5$, the structured layer returns $w=0.45w_{\text{Kelly}}$. This is an illustrative decomposition showing how independent effects stack multiplicatively; it is not yet a production-ready robo-advice rule.

6.3 Pension Fund Governance

The effective horizon concept (Section 5.5) provides a specific governance lens for pension settings. A pension fund with ongoing contributions and a young beneficiary population routinely exhibits an effective horizon $T_{\text{eff}} \gg T_{\text{own}}$ far exceeding any individual beneficiary's lifespan. In our extension layer, this mismatch mathematically pushes the optimal allocation strictly closer to the long-horizon benchmark. This is a structural modeling lens, not a direct governance recommendation.

6.4 Crisis Response

The regime shift theorem (Proposition 8) reveals a counter-intuitive comparative static: faster mean reversion strictly raises the harvestability-driven correction term. This does not blindly force a "buy more equity in every crisis" prescription. It proves only that, holding all other parameters fixed, the horizon-dependent correction becomes mathematically more favorable when the mean-reversion speed increases.

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7. Machine Verification: Lean 4 Kernel Proofs

7.1 Why Machine Verification?

The literature on horizon-dependent allocation contains numerous conflicting claims, sign errors, and imprecise arguments. Campbell and Viceira (1999) use log-linear approximations; Wachter (2002) derives closed forms under specific assumptions; Barberis (2000) uses numerical methods. None of these results are formally verified.

I formalize the structural backbone of the theory in a Python-native proof language whose every statement is verified by a formal proof kernel and is exportable to Lean 4. The kernel verifies 141 theorems covering core harvestability properties, monotonicity, the first-principles derivation chain, and lifecycle extensions — with **zero unproved statements (0 sorries)**.

7.2 Proof Architecture

The verified proofs live in three proof files:

- elysium/fields/harvestability/harvestability_proof.py (63 theorems: core properties, monotonicity, bounds, Merton recovery, spectral shortfall with its order-four complete-monotonicity ladder, the proved single-factor approximation bound, the dispersion origin-rise sign, the dynamic real-time harvestability object — bounded, monotone in the integrated harvest, sandwiched by static h, and concave with a kernel-verified two-sided mean-path bracket — and the real-eigenvalue structure of the two-regime exact value function)
- elysium/fields/harvestability_derivation/harvestability_derivation_proof.py (58 theorems: the CRRA/OU/HJB/Riccati derivation chain and comparative statics)
- elysium/fields/harvestability_extensions/harvestability_extensions_proof.py (20 theorems: the structured lifecycle layer)

The theorems are organized across three domains: core harvestability properties, the first-principles derivation chain, and the downstream lifecycle extensions.

7.3 Verification Statistics

Metric	Value
Proof files	3
Kernel-verified statements	141 (63 core + 58 derivation + 20 lifecycle)
Unproved statements (sorries)	**0**
Kernel errors	**0**
Vacuous / tautological / delegation statements	**0**
Foundation	Mathlib-backed real-analysis facts (Real.exp_pos, Real.exp_le_exp, Real.div_pos, Real.mul_pos, …) declared via p.mathlib
Residual derivation-debt	**0** (and 0 domain axioms) — all theorems fully derived from the Mathlib foundation; the Merton-limit bound exp_lt_of_lt_log is proved (§4.7), not axiomatized
Export target	Lean 4 (every kernel statement is Lean-exportable)

Claim grade breakdown. To keep the count honest, every kernel statement is graded by the content its proof actually establishes (substantive / structural / definitional), and the count is audited for the failure modes that let a proof assistant compile a theorem with no mathematical content — tautologies whose proof is a hypothesis, aliases that merely re-call another theorem, and vacuous witnesses that satisfy the conclusion by construction while ignoring the hypotheses. None are present here.

Grade	Meaning	Core	Derivation	Lifecycle	Total
A — Substantive	conclusion derived from non-trivial hypotheses by tactic (nlinarith / linarith /ring, monotonicity of exp)	34	56	20	**110**
B — Structural	assembles other verified theorems (e.g. Merton recovery composes the residual identity with the exp decay bound)	25	2	0	**27**
C — Definitional	closed-form–unfolding identities (fh_def, hw_def, ea_def, bcf_def) — trivial by design (§4.5)	4	0	0	**4**
D/E/F — Tautology / Delegation / Vacuous	proofs with no mathematical content	0	0	0	**0**
	Total	**63**	**58**	**20**	**141**

So of the 141 kernel statements, **137 are substantive or structural theorems** (grades A + B) and 4 are definitional identities (grade C); there are **no** tautologies, aliases, or vacuous witnesses. Every result cited in Sections 3–5 and in Appendix A resolves to a named theorem in one of these three files (verified by a name-level consistency check against the source). The proofs rest on standard real-analysis facts drawn from Mathlib rather than on domain-specific axioms. Some surrounding analytic estimates and economic interpretations are stated as classical prose and are

explicitly flagged as *model-level* or *classical* rather than as kernel theorems, so the proof boundary is never blurred.

7.4 What Machine Verification Buys

1. **No sign errors**: The hedging coefficient $(\gamma - 1)/\gamma^2$ has the correct sign for conservative ($\gamma > 1$) and aggressive ($\gamma < 1$) investors. Getting this wrong would reverse the entire conclusion — and published papers have gotten it wrong.
2. **Correct boundary conditions**: The Riccati terminal condition $\alpha(T) = 0$ is verified, not assumed. The harvestability function is $h(T - t, \tau)$, not $h(t, \tau)$ — a common source of confusion.
3. **Quantifier precision**: "For all $T > 0$ and $\tau > 0$" is enforced. No hidden assumptions.
4. **Algebraic consistency of the proof chain**: once the closed-form ODE solution is written through h, the allocation decomposition and residual identities line up without algebra error (hd_w11_allocation_decomposition, hd_w11_samuelsonError_eq_residual, hd_w11_derivation_agrees_with_assumption). These are definitional/bookkeeping checks, not a machine proof of the ODE solve itself — that step is classical and is confirmed numerically (§7.5). The point of formalizing them is that the chain from closed form to decomposition to Samuelson residual is guaranteed self-consistent.

7.5 Numerical Validation

Formal proof and numerical simulation validate different things, and the paper provides both. The kernel proofs certify the *algebra* (bounds, monotonicity, sign of the hedging correction, limits); the companion script numerical_validation.py certifies the *modeling claim* the algebra cannot reach — that $h(T, \tau)$ is genuinely the harvestable fraction of an Ornstein–Uhlenbeck premium — via Monte-Carlo simulation, and independently confirms the closed-form Riccati solution by numerical ODE integration. Results are reported in Appendix B.4: the Monte-Carlo estimate matches $h(T, \tau)$ to within 10^{-3}, and the numerical ODE solution matches the closed form to $\sim 10^{-14}$. Every illustrative table in Appendix B is regenerated and asserted by the same script, so the paper's numbers cannot drift from the model. The framework thus rests on three mutually reinforcing legs: **formal proof** (141 kernel theorems), **numerical validation** (Monte-Carlo + ODE + spectrum recovery, this section), and **out-of-sample empirical evidence** on real Fama–French data (Section 8, backtest.py), with calibration tooling in ndvr_harvestability_calibration.py.

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8. Empirical Evidence on Real Data

The formal and numerical layers verify the *internal* structure of the theory. This section asks the *external* question: does harvestability-based mode selection improve outcomes on real return data, not only on data generated to obey the model? The answer is a qualified yes, and the qualification is itself informative.

8.1 Setup

We use the Fama–French 48 Industry Portfolios (value-weighted monthly returns, CRSP, 1926–2026; 40 industries with complete history) with the risk-free rate from the Fama–French 5-factor file. On an expanding, out-of-sample window we PCA the industry covariance into its top five eigenmodes. Each mode k carries a variance σ_k^2 (eigenvalue), a premium ρ_k (mean score), and a mean-reversion time scale τ_k estimated as the AR(1) half-life of the mode's *detrended cumulative* score — a valuation-level proxy, since monthly returns are near-white and the mean-reverting structure lives in the slow-moving level (consistent with the predictability literature — Campbell and Viceira 1999 likewise use valuation ratios, i.e. integrated returns, as the predictor state). Weights are re-estimated at each annual rebalance using only past data. We simulate overlapping 30-year investor cohorts ($\gamma = 3$) under three strategies:

- **Merton** — constant myopic mode weights $w_k \propto \rho_k/(\gamma\sigma_k^2)$, fully invested (Samuelson horizon-independence).
- **Harvestability** — the same weights scaled by $h(T-t, \tau_k) = 1 - e^{-(T-t)/\tau_k}$ and renormalized, fully invested. This is a *pure mode tilt*: identical risk budget to Merton, differing only in which modes are emphasized.
- **Naive glide** — the industry "100-minus-age" rule: risky fraction declines linearly with age, de-risking into the real risk-free asset.

Holding Merton and Harvestability at the same total risk budget isolates the paper's actual mechanism — *which* modes to harvest — from the orthogonal risk-level (cash) decision.

Scope of the tested object. The tilt implemented here is the *multiplicative* benchmark-scaled surface $w_k^{\text{Merton}}\, h(T-t, \tau_k)$ — the auxiliary object quarantined in §3.4 — not the canonical *additive* allocation $w_k^{\text{Merton}} + \eta(\gamma)(\rho_k/\sigma_k^2)\, h(T-t, \tau_k)$ of §4.6. The two share the *sign* of the horizon correction (down-weight still-slow modes as the horizon shortens), so this backtest is an out-of-sample test of the mechanism's *direction*, not of the exact additive decomposition. The multiplicative form is chosen deliberately: it keeps the total risk budget fixed, which is what isolates the mode-selection channel from the risk-level channel.

8.2 Results

Strategy	Mean CE (ann.)	Median CE	Sharpe	Harvest CE win-rate
Merton (constant)	0.57%	0.60%	0.137	—
Naive glide (100−age)	2.59%	2.87%	0.872	Harvest wins 8%
Harvestability tilt	**0.68%**	**0.65%**	**0.157**	vs Merton: **80%**

*(Certainty-equivalent returns are conservative CRRA quantities on a pure-industry universe; the relevant content is the cross-strategy comparison, reproducible via topics/fin_harvestability/backtest.py. The **80%** figure is 20 of 25* overlapping* cohorts — only ≈ 2.7 independent 30-year windows, HAC $t \approx 1.8$; read it as a directional sign, not a significance level. See point 1.)*

Two findings, one positive and one cautionary:

1. **The mode-tilt mechanism works, modestly and directionally.** At an identical risk budget, the harvestability tilt beats constant Merton in **20 of 25 overlapping 30-year cohorts** (+11 bps/yr mean CE, Sharpe 0.157 vs 0.137). Down-weighting still-slow modes as the horizon shortens trims risk faster than it trims return — exactly the sign the theory predicts. **This win-rate must be read with care.** The 30-year cohorts are spaced 24 months and overlap by $\approx 93\%$, so the century-long sample holds only ≈ 2.7 *independent* 30-year windows. Under Newey–West HAC inference (lag set to the 15-cohort overlap) the mean per-cohort edge carries $t \approx 1.8$ (two-sided $p \approx 0.07$), and a moving-block bootstrap gives a wide 90% band on the win-rate. The defensible reading is therefore the *sign and small magnitude* of the edge — consistently positive across cohorts — not a powered win-rate statistic. The effect is small because monthly equity mean-reversion is weak.
2. **The naive glide's apparent superiority is a risk-free-timing artifact, not portfolio skill.** The "100-minus-age" rule posts the highest CE and Sharpe only because it parks a large, growing share of wealth in Treasury bills during the high-rate decades (notably the 1980s). This is a *risk-level* effect that has nothing to do with mode selection, and it disappears the moment one controls for the cash decision (which is why the mechanism test holds risk budget fixed). Read correctly, the result reframes the industry rule: its historical performance came from a fortuitous cash allocation, not from a principled glide shape.

8.3 Honest scope

This is a proof-of-concept on real data, not a definitive empirical claim. The signal is modest and sample- and parameter-dependent (choice of γ, mode count, and the level-based τ estimator). In particular the 30-year cohorts overlap heavily (≈ 2.7 independent windows in a century of data), so the cohort win-rate is a *descriptive*, not an *inferential*, statistic; the defensible empirical claim is the consistently-signed +11 bps/yr edge (Newey–West HAC $t \approx 1.8$, $p \approx 0.07$), which is directional rather than significant at conventional levels. What it does establish is that the paper's specific contribution — the harvestability mode tilt — is *directionally correct out-of-sample on real returns*, and that a naive comparison against the industry glide rule can mislead unless the risk-level and mode-selection channels are separated. The backtest is fully reproducible and reported without cherry-picking cohorts.

8.4 Model-Free Evidence for Multi-Timescale Structure

The backtest tests the allocation rule; this subsection tests the paper's *theoretical* prediction directly and without a portfolio model. Section 3.3.2 predicts that a genuine premium spectrum has more than one time scale, so a single-timescale model must be misspecified. The classical model-free probe of horizon structure is the variance ratio $\mathrm{VR}(k) = \mathrm{Var}(r_{t:t+k})/(k\,\mathrm{Var}(r_t))$ (Poterba and Summers, 1988): $\mathrm{VR} > 1$ signals momentum, $\mathrm{VR} < 1$ mean reversion. The key observation is structural: **a single mean-reversion time scale yields a monotone VR term structure that never crosses unity** — it is above 1 everywhere for a momentum mode, below 1 everywhere for a reversion mode. A term structure that *crosses* unity therefore requires at least two time scales.

The real equity market crosses unity, on both proxies:

Horizon k (months)	VR (EW industries, 1926–2026)	VR (VW market, 1963–2026)
3	1.15	1.02

Horizon k (months)	VR (EW industries, 1926–2026)	VR (VW market, 1963–2026)
12	1.10	1.06
24	0.99	0.99
36	0.82	0.86
60	0.61	0.86
120	0.31	0.89

Both series show short-horizon momentum (VR > 1) turning into long-horizon mean reversion (VR < 1) — precisely the two-timescale signature (a fast positive-autocorrelation mode plus a slow reverting mode), and consistent with two separately documented stylized facts (short-horizon momentum; long-horizon reversion). No single τ can produce both signs. This is the empirical counterpart of the implied-timescale drift of Section 3.3.2: the effective time scale a single-factor model would infer changes with the horizon.

Honest scope. The sign pattern is robust across the two proxies, but the long-horizon magnitudes are imprecise — with a century of data there are few independent decade-long windows. Under a block bootstrap that destroys long memory, the equal-weight VR(120) $= 0.31$ is more extreme than $\approx 99\%$ of resamples (a rough one-tailed significance of ≈ 0.01), while the value-weighted reversion is milder and weaker. The claim is deliberately modest: the *shape* (a crossing of unity) is the model-relevant fact, and it is present in real data. Reproducible via backtest.py (results/variance_ratio_term_structure.md).

8.5 Measuring the Spectrum and Rejecting a Single Time Scale

The variance-ratio crossing shows that *more than one* time scale is present. Two sharper questions follow, and both are answerable with the machinery the theory makes identifiable (Section 3.3.2): *what* is the time-scale spectrum of the real premium, and can the *single*-time-scale hypothesis be formally rejected?

Estimating the τ-spectrum (Laplace inversion). Because the multi-mode reading makes the OU-state autocovariance a nonnegative mixture of exponentials — $\rho(j) = \sum_k w_k\, e^{-j/\tau_k}$ with $w_k \geq 0$, a completely monotone sequence — the spectrum is a discrete Laplace transform of a positive measure and can be recovered by nonnegative least squares on a log-spaced τ-grid (the stable, real-data analogue of the clean-data Prony demonstration in Appendix B.5). Applied to the century-long equal-weight market, the inversion returns a genuinely multi-modal spectrum: a participation ratio of ≈ 2.5 effective modes, with mass concentrated in a fast band near $\tau \approx 1.5$–1.7 years (weight ≈ 0.84) and a slow band of several years (weight ≈ 0.16). A single mean-reversion time scale would place all mass on one node (participation ratio ≈ 1). This is a *measured* object about the equity premium, not a calibration input — the empirical realization of the paper's τ-spectrum. One caveat on the precise weights: τ_k is estimated from the *detrended cumulative* score, and that transform can inject low-frequency power, so the slow band's mass (≈ 0.16) is best read as an *upper* estimate rather than a sharp point value. The robust, transform-independent conclusion is the qualitative one — a participation ratio strictly above 1 (multi-modal), not a single node. On the shorter (60-year) value-weight sample the inversion is under-determined and collapses to a single band with a markedly worse fit, so the spectrum resolves cleanly only where the history is long.

Rejecting the single-time-scale null (specification test). Proposition 3b states that a completely monotone term structure is convex on every horizon grid; a single time scale therefore produces a *monotone* variance ratio that cannot cross unity. We turn this into a formal test. The statistic is the crossing signature min(short-horizon excess, long-horizon deficit), which is strictly positive only when short-horizon momentum (VR > 1) and long-horizon reversion (VR < 1) coexist — the two-mode fingerprint. The null gives the single-time-scale model its best shot: the AR(1) parameter is fit to minimize squared error against the *whole* observed VR curve, and surrogate paths are generated by parametric bootstrap. In the equal-weight market the best single time scale cannot reproduce the observed crossing in any of 5,000 surrogates ($p < 0.001$); the two-mode signature is not sampling noise around a one-time-scale process. One caveat on interpretation: an AR(1) null is *structurally* unable to cross unity, so the test formalizes "the data crosses and a single-time-scale process essentially never does" — it is a specification test against the one-mode class, not a horse race between two comparably flexible models. Its force comes from the crossing being present in the data (the VR table above) and impossible under the null, not from a fine calibration contest.

The value-weight market, by contrast, is only marginal ($p \approx 0.06$), and a three-way panel isolates why. Two confounds could explain the equal-weight/value-weight gap: the value-weight sample is shorter (post-1963), and value-weighting itself downweights small-capitalization names. Running the *same* test on the equal-weight market **restricted to the value-weight window** (1963–2026) is the control that separates them — and it still rejects decisively ($p < 0.001$). The gap is therefore a **weighting effect, not a sample-length artifact**: the long-horizon reversion that forces a second time scale is concentrated in the smaller-cap names that value-weighting suppresses. This is itself an economically sensible finding, consistent with the well-documented size tilt of long-horizon return predictability, and it converts the value-weight weakness from an unexplained gap into a mechanism.

Together these convert the paper's spectral machinery from interpretation into two empirical results: a *measured* premium time-scale spectrum, and a *formal rejection* of the single-time-scale specification that is robust across the long sample and the equal-weight scheme, with the value-weight marginality explained by the size concentration of long-horizon reversion. Reproducible via backtest.py (results/tau_spectrum.md, results/single_mode_test.md).

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9. Conclusion

This paper establishes harvestability as a proof-oriented horizon object for portfolio allocation. Its central contribution does not claim long-horizon economics were previously absent. Instead, it defines the object cleanly, derives it from an explicit OU/HJB/Riccati spine, and separates the core result from downstream extensions in a machine-checkable way.

The harvestability function $h(T, \tau) = 1 - e^{-T/\tau}$ provides the exact, closed-form correction term. We derived it from the Hamilton–Jacobi–Bellman equation via a Riccati ODE; it is never assumed. Under this formulation, the myopic Merton allocation remains the benchmark, and harvestability organizes the horizon-dependent correction around it. Consequently, the Samuelson Error $\varepsilon(T, \tau) = e^{-T/\tau}$ emerges as a derived consequence that quantifies exactly how much a finite-horizon investor fails to capture relative to the limit.

The extended lifecycle assembly $w = w_{\text{Kelly}} \cdot \mathbb{E}[h] \cdot c_{\text{Bayes}} \cdot s_{\text{safety}}$ shows one disciplined way to scale the root object into a broader lifecycle filter involving bequest motives, stochastic mortality, parameter

uncertainty, and wealth-floor constraints. This is a structured downstream package, not a claim to have solved the entire lifecycle problem in one economic specification.

The true value of this manuscript lies in its role as a foundation for the research program. Benchmark-side papers can safely import the investor-specific harvesting layer, while calibration papers can reuse the object and its proof boundary without re-establishing the theory.

The dynamic, real-time extension of harvestability through the Bayesian Live Risk framework (Nagy, 2026) is now partially resolved (§4.9). In BLR, a sequential Monte Carlo filter tracks spectral coefficients in real time, producing a posterior width U_t (model uncertainty) and a crisis memory state M_t; a wide posterior or active memory ($M_t > 0$) signals that mode premiums are uncertain and less harvestable, and the memory-aware capital buffer $B_t^{\text{memory}} = B_{\text{base}} + \lambda U_t + \eta M_t$ rises accordingly. Section 4.9 embeds this signal into the derivation as a time-varying Riccati coefficient $\kappa(s) = g(U_s, M_s)/\tau$: the linear ODE still integrates exactly, and the harvestable fraction generalizes to the integrated-exponential $h_{\text{dyn}} = 1 - e^{-\int g/\tau}$, which recovers the static object in calm markets and is kernel-verified to move with the right sign under stress.

The exact form holds when the confidence path is deterministic (a known or forecast trajectory — the certainty-equivalent reading); for a stochastic real-time signal with a general CRRA investor ($\gamma \neq 1$) the dynamic object is rigorously *sandwiched* between two static harvestabilities rather than solved in closed form. That sandwich is not static: the concavity of $1 - e^{-I}$ is now kernel-verified on *both* sides (dyn_harvest_tangent_upper above, dyn_harvest_chord_lower below), sharpening the crude constant-g sandwich to a two-sided **mean-path bracket** $\text{chord}(\mathbb{E}[I]) \leq \mathbb{E}[h_{\text{dyn}}] \leq 1 - e^{-\mathbb{E}[I]}$ whose edges depend on the mean integrated harvest rather than on worst/best-case constants. In the stochastic crisis simulation this narrows the enclosure from the crude $[0.731, 0.977]$ to $[0.805, 0.871]$ — about 3.7× tighter around the true $\mathbb{E}[h_{\text{dyn}}] = 0.870$ (§4.9, Appendix B.6). Notably the chord bound stays inside the kernel's algebra+exp layer (a convex function is the supremum of its tangents, so the chord follows from Real.add_one_le_exp applied at the convex-combination point — no native analysis needed). Beneath the bracket, the fully-coupled $\gamma \neq 1$ value function is then solved *exactly* in the natural discrete reduction of the signal (§4.10): when crisis memory is read as a regime, the confidence path becomes a continuous-time Markov chain and the value function is a Markov-modulated matrix exponential — a genuine bi-exponential in the two-regime case, with the real-eigenvalue structure kernel-verified (regime_generator_real_eigenvalues) and the frozen-regime limit recovering the sandwich endpoints. The bracket is moreover confirmed to enclose the *actual* $\gamma \neq 1$ value function, not merely the linear proxy $\mathbb{E}[h_{\text{dyn}}]$: because the confidence signal is a Bayesian belief rather than a separately tradeable factor, the dominant γ-effect is the nonlinear risk aggregation of the random harvest rather than a hedging demand against a traded g, and the value function is the CRRA certainty-equivalent of the random harvest, which a Monte-Carlo estimate and an independent HJB/Feynman-Kac PDE solve both place inside the bracket with a negligible risk correction (§4.9, Appendix B.6).

The residual that remains open is therefore narrowed to a single object: the exact *elementary* closed form in the *continuous, non-affine* case — a bounded signal multiplicatively modulating the reversion rate — where no elementary form survives. Even this residual is now quantitative: a small-noise expansion supplies the exact leading-order value $1 - e^{-\mathbb{E}[I]} - \frac{1}{2}e^{-\mathbb{E}[I]}\text{Var}(I)$ with $O(\sigma^3)$ error (closed-form variance for an Ornstein–Uhlenbeck signal, verified 65–400× more accurate than the Jensen edge), and a matching variance-corrected bracket tightens the enclosure further (§4.9). The kernel-verified mean-path bracket, now known to bound the true value function and pinned to a point estimate in the small-noise regime, is the sharpest rigorous statement.

Declaration of Generative AI and AI-assisted technologies in the writing process

During the preparation of this work the author used Cursor and its integrated AI models (Claude and GPT families) in order to assist with manuscript drafting, structural editing, and language polishing. After using this tool/service, the author reviewed and edited the content as needed and takes full responsibility for the content of the publication.

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Appendix A: Formal Proof Map

The following table maps the paper's formalized structural backbone to its Lean 4 kernel theorem. All theorems are verified in the three proof files listed in Section 7.2. It is a proof locator, not a claim that every surrounding analytic estimate or interpretation sentence in the prose is itself formalized.

Paper Result	Formal Theorem	Proof File
Definition 1 (Harvestability)	fh_def	harvestability_proof.py
Proposition 1a ($0 < h < 1$)	harvestability_bounded	harvestability_proof.py
Proposition 1b ($h(0) = 0$)	harvestability_at_zero	harvestability_proof.py
Proposition 1c ($h \to 1$)	harvestability_residual	harvestability_proof.py
Proposition 1d (increasing in T)	harvestability_increasing_in_T	harvestability_proof.py
Proposition 1e (decreasing in τ)	harvestability_decreasing_in_tau	harvestability_proof.py
Proposition 1f ($h \leq T/\tau$)	harvestability_le_linear	harvestability_proof.py
Proposition 2a ($h + \varepsilon = 1$)	harvestability_plus_error_is_one	harvestability_derivation_proof.py
Proposition 3 (mode ordering)	fast_modes_more_harvestable	harvestability_proof.py
§3.3.1 (slow modes dominate residual)	slow_mode_dominates_shortfall	harvestability_proof.py
§3.3.1 (spectral shortfall positive)	spectral_shortfall_pos	harvestability_proof.py

Paper Result	Formal Theorem	Proof File
§3.3.1 (shortfall below total myopic)	spectral_shortfall_lt_total	harvestability_proof.py
§3.3.1 (dispersion vanishes at origin)	harvest_dispersion_zero_at_origin	harvestability_proof.py
§3.3.1 (dispersion rises from origin, $D'(0) > 0$)	dispersion_balance_positive_at_origin	harvestability_proof.py
§3.3.2 (shortfall strictly decreasing, $S' < 0$)	spectral_shortfall_strictly_decreasing	harvestability_proof.py
§3.3.2 (shortfall strictly convex, $S'' > 0$)	spectral_shortfall_convex	harvestability_proof.py
§3.3.2 (third derivative, $S''' < 0$)	spectral_shortfall_third_deriv_neg	harvestability_proof.py
§3.3.2 (fourth derivative, $S'''' > 0$)	spectral_shortfall_fourth_deriv_pos	harvestability_proof.py
Proposition 3b (cross-horizon consistency test)	spectral_term_structure_discrete_convex	harvestability_proof.py
Proposition 7a (glide path decreasing)	glide_path_decreasing	harvestability_proof.py
Proposition 8 (crisis comparative static)	crisis_increases_allocation	harvestability_proof.py
Proposition 9 (Bayesian correction)	bayesian_correction_reduces_weight	harvestability_proof.py
§4.9 (dynamic h_{dyn} bounded, $0 < 1 - e^{-I} < 1$)	dyn_harvest_pos, dyn_harvest_lt_one	harvestability_proof.py
§4.9 (dynamic comparative static, monotone in I)	dyn_harvest_monotone_in_integral	harvestability_proof.py
§4.9 (sandwich building block, $I_1 \leq I_2$)	dyn_harvest_le_of_integral_le	harvestability_proof.py
§4.9 (concavity / mean-path Jensen upper engine)	dyn_harvest_tangent_upper	harvestability_proof.py
§4.9 (concavity / mean-path chord lower engine)	dyn_harvest_chord_lower	harvestability_proof.py
§4.10 (regime-switching real eigenvalues / bi-exponential)	regime_generator_real_eigenvalues	harvestability_proof.py
Theorem 2 (derivation, hedging = harvestability)	hd_w11_riccati_solution_is_harvestability	harvestability_derivation_proof.py
Theorem 4 (recovery bundle)	full_lifecycle_theorem	harvestability_extensions_proof.py
Theorem 5 (full lifecycle)	full_lifecycle_theorem	harvestability_extensions_proof.py

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Appendix B: Numerical Illustrations

All tables in this appendix are reproducible and machine-checked by the companion script topics/fin_harvestability/numerical_validation.py, which also validates the modeling claim by Monte-Carlo simulation of the underlying Ornstein–Uhlenbeck process (the analytic $h(T, \tau)$

matches the simulated harvested fraction to within 5×10^{-3}) and confirms the closed-form Riccati solution by independent numerical integration (agreement to $\sim 10^{-14}$).

B.1 Harvestability Curves

For a mode with $\tau = 5$ years (typical equity mean-reversion time scale):

Horizon T (years)	$h(T, 5)$	$\varepsilon(T, 5)$	Interpretation
1	0.181	0.819	Short-term trader: harvests 18%
5	0.632	0.368	Medium-term: harvests 63%
10	0.865	0.135	Long-term: harvests 86%
20	0.982	0.018	Very long: nearly full-harvest
30	0.998	0.002	Pension fund: effectively full-harvest

B.2 Multi-Mode Allocation

For a two-mode portfolio with $\tau_1 = 2$ years (fast) and $\tau_2 = 10$ years (slow), and Merton weights $w_1^M = 0.4$, $w_2^M = 0.6$:

Horizon	$w_1^*(T)$	$w_2^*(T)$	Total
1 year	0.157	0.057	0.214
5 years	0.367	0.236	0.603
10 years	0.397	0.379	0.777
20 years	0.400	0.519	0.919
30 years	0.400	0.570	0.970

The fast mode is nearly fully harvested by year 10; the slow mode requires 30+ years. This illustrates the pecking order.

B.3 Glide Path Example

For a CRRA investor with $w_{\text{Merton}} = 0.6$, $\tau = 5$ years, retirement age $R = 65$:

Age	Horizon $(R - a)$	$h(R - a, 5)$	Equity w^*
25	40	0.9997	60.0%
35	30	0.9975	59.9%
45	20	0.9817	58.9%
55	10	0.8647	51.9%
60	5	0.6321	37.9%
63	2	0.3297	19.8%

Age	Horizon $(R-a)$	$h(R-a,5)$	Equity w^*
64	1	0.1813	10.9%
65	0	0.0000	0.0%

The exponential glide path is nearly flat for most of the working life, then drops sharply near retirement — a shape that is qualitatively similar to, but more principled than, typical target-date fund designs.

B.4 Numerical Validation Results

The formal kernel proofs verify the *algebra* of harvestability. The companion script numerical_validation.py additionally verifies the *modeling claim* — that the analytic $h(T,\tau)$ is the harvestable (mean-reverting) fraction of the underlying Ornstein–Uhlenbeck premium — by direct Monte-Carlo simulation (2×10^5 paths, daily steps, fixed seed). For an OU process $dX=-(X/\tau)\,dt+\sigma\,dW$, the conditional mean reverts as $\mathbb{E}[X_T\mid X_0]=X_0\,e^{-T/\tau}$, so the reverted fraction $1-\mathbb{E}[X_T]/X_0$ equals $h(T,\tau)$. Simulation confirms this:

τ	T	h (analytic)	h (simulated)	abs. error
2	1	0.3935	0.3942	0.0008
2	5	0.9179	0.9187	0.0008
2	10	0.9933	0.9937	0.0004
5	1	0.1813	0.1821	0.0009
5	5	0.6321	0.6326	0.0005
5	10	0.8647	0.8649	0.0002
10	1	0.0952	0.0954	0.0003
10	5	0.3935	0.3937	0.0003
10	10	0.6321	0.6330	0.0009

Maximum absolute error across all cases is below 10^{-3} (Monte-Carlo noise). Independently, the closed-form Riccati solution $\alpha(t)=C\tau\,(1-e^{-(T-t)/\tau})$ is confirmed by backward RK4 integration of its ODE $\dot\alpha=\alpha/\tau-C,\ \alpha(T)=0$, with maximum deviation $\sim 2\times10^{-14}$. Finally, the script regenerates every table in Appendix B and asserts agreement with the printed values, so the paper's illustrations cannot silently drift from the model. All checks pass.

B.5 The Spectral Reading: Recovery, Implied-Timescale Drift, and Approximation Cost

Three numerical demonstrations back the completely-monotone/Laplace reading of §3.3.2. All are checked by numerical_validation.py.

Spectrum recovery (Prony). From eight equally-spaced samples of a known two-mode term structure $S(T)=0.35\,e^{-T/2.5}+0.65\,e^{-T/12}$, Prony's method recovers the full spectrum — both time scales and both amplitudes — to a maximum error of $\sim10^{-13}$. The premium time-scale spectrum is thus *identifiable* from the horizon term structure, not merely a heuristic.

Implied-timescale drift. For a two-mode spectrum ($\tau_1 = 2$, $\tau_2 = 10$, $a = 0.4$, $b = 0.6$), the single effective timescale a horizon-T investor infers, $\tau_{\text{eff}}(T) = -T/\ln(S(T)/(a+b))$, drifts strictly upward:

Horizon T (years)	$S(T)$	$\tau_{\text{eff}}(T)$ (years)
0.5	0.882	3.99
2	0.638	4.46
5	0.397	5.41
10	0.223	6.67
30	0.030	8.55

A flat implied timescale would signal a single mode; the drift from 3.99y to 8.55y is the fingerprint of a genuine spectrum — a testable analog of the implied-volatility smile.

Cost of collapsing the spectrum. A practitioner who fits one timescale at a short horizon ($T_0 = 1$y, giving $\hat{\tau} = 4.14$y) then applies it across horizons misprices the harvestable fraction by up to **18% in relative terms** over a 0.5–30y grid. The exact continuous closed form does not merely claim to be exact; it *measures* the error of the single-persistence approximations used downstream.

Consistency discriminator (Proposition 3b). The discrete second-difference test discriminates: a genuine two-mode term structure ($\tau_1 = 2, \tau_2 = 10$) passes on a 1–25y grid (all second differences > 0), while a fabricated concave curve $S = 1 - 0.0008\,T^2$ is flagged as inconsistent (minimum second difference $-1.6 \times 10^{-3} < 0$) — it is a Laplace transform of no positive spectrum. All checks pass.

B.6 Dynamic (Real-Time) Harvestability

Eleven numerical demonstrations back the dynamic extension of §§4.9–4.10, on a horizon $T = 30$y with $\tau = 8$y and a Gaussian crisis dip in the confidence signal $g(s) = 1 - 0.5\,e^{-(s-12)^2/(2\cdot 3^2)}$ (so $g_{\min} = 0.5$). All are checked by numerical_validation.py.

Integrating-factor exactness. A direct backward solve of the time-varying Riccati ODE $\alpha'(t) = \kappa(t)\alpha(t) - C$, $\alpha(T) = 0$, with $\kappa = g/\tau$, agrees with the closed integrating-factor form $\alpha(0) = C\int_0^T e^{-\int_0^u \kappa}\,du$ to a relative error of 5.4×10^{-6}. The time-varying ODE genuinely integrates to the integrated-exponential $1 - e^{-I}$.

Calm recovery. Setting $g \equiv 1$ gives $I(t) = (T-t)/\tau$, and the dynamic object equals the static $h(T-t, \tau)$ across the whole grid to a maximum error of 0 (machine precision).

Sandwich. With the crisis path, the dynamic harvestability is trapped between the two static bounds at every horizon; at $t = 0$, $h(30, \tau/g_{\min}) = 0.847 \le h_{\text{dyn}} = 0.962 \le h(30, \tau) = 0.977$.

Stress monotonicity. Deepening the crisis dip (from 0.5 to 0.8, i.e. shrinking the integrated harvest) strictly lowers $h_{\text{dyn}}(0)$ from 0.962 to 0.950 — the comparative static of dyn_harvest_monotone_in_integral.

Stochastic mean-path Jensen bound. For the fully-coupled case — a genuine stochastic confidence signal, simulated as a reflected mean-reverting (Ornstein–Uhlenbeck) stress process with a floor, 40,000 paths — the Monte-Carlo expected dynamic harvestability is $\mathbb{E}[h_{\text{dyn}}] = 0.870$, bounded above by the mean-path Jensen edge $1 - e^{-\mathbb{E}[I]} = 0.871$ (kernel engine dyn_harvest_tangent_upper)

with a slack of only 0.002. That edge sits far below the crude calm-market pathwise upper bound $h(T,\tau) = 0.977$, confirming the mean-path bound is a genuine tightening, not a restatement.

Concavity (tangent) bound. The pointwise inequality $1 - e^{-x} \le (1 - e^{-m}) + e^{-m}(x - m)$ underlying the Jensen step holds across 5,000 random (x, m) pairs on $[0, 6]^2$ (maximum violation -1.7×10^{-8}, i.e. satisfied).

Mean-path lower edge and the bracket. Applying the concave-chord bound along the pathwise range $I \in [a, b]$ ($a = g_{\min}T/\tau$, $b = T/\tau$) tightens the crude pathwise lower edge from $h(a) = 0.731$ to the mean-path chord value 0.805. Combined with the Jensen upper edge, the two-sided mean-path bracket [0.805, 0.871] encloses the simulated $\mathbb{E}[h_{\text{dyn}}] = 0.870$ about 3.7× more tightly than the crude sandwich [0.731, 0.977].

Concavity (chord) bound. The pointwise inequality $1 - e^{-(\lambda a + (1-\lambda)b)} \ge \lambda(1 - e^{-a}) + (1-\lambda)(1 - e^{-b})$ underlying the lower edge holds across 5,000 random (a, b, λ) triples on $[0, 6]^2 \times [0, 1]$ (maximum violation -3.5×10^{-10}, i.e. satisfied).

Regime-switching exact value function (§4.10). For the discrete-confidence case — a two-state calm/crisis Markov chain with harvest levels $(g_{\text{calm}}, g_{\text{crisis}}) = (1.0, 0.5)$ and transition intensities $(\lambda_{\text{c}\to\text{r}}, \lambda_{\text{r}\to\text{c}}) = (0.4, 0.8)$ — the closed-form Markov-modulated matrix exponential $h_{\text{dyn},i} = 1 - (e^{(T-t)(Q - \text{diag}(g)/\tau)}\mathbf{1})_i$, computed via spectral projectors, matches a 60,000-path Monte-Carlo simulation of the chain to a maximum error of 3.7×10^{-5}. The generator discriminant is $\Delta = 1.39 > 0$ (real eigenvalues, decay rates $\mu = -0.103, -1.284$), so the exact value function is a genuine bi-exponential (regime_generator_real_eigenvalues). The frozen-regime limit $Q \to 0$ reproduces the static harvestability $h(T,\tau) = 0.977$ to 1.4×10^{-6}, and the exact values $(h_{\text{calm}}, h_{\text{crisis}}) = (0.956, 0.953)$ lie inside the static sandwich $[0.847, 0.977]$.

Value-function enclosure (§4.9). To confirm the bracket bounds the actual $\gamma \neq 1$ value function rather than only the linear proxy $\mathbb{E}[h_{\text{dyn}}]$, the CRRA certainty-equivalent harvestable fraction $h_\gamma = (\mathbb{E}[(1 - e^{-I})^{1-\gamma}])^{1/(1-\gamma)}$ is computed for a clipped Ornstein–Uhlenbeck belief signal ($\bar{g} = 0.85$, $\theta = 0.6$, $\sigma_g = 0.25$, bracket $[0.892, 0.945]$) two independent ways: a 60,000-path Monte-Carlo estimate and a genuine backward finite-difference solve of the HJB/Feynman–Kac PDE $M_t + \theta(\bar{g} - g)M_g + \frac{1}{2}\sigma_g^2 M_{gg} + (g/\tau)M_y = 0$ on the augmented state (g, y). For an aggressive investor ($\gamma = 0.5$) both give $h_\gamma \approx 0.944$–0.938 and for a conservative one ($\gamma = 5$) ≈ 0.945–0.936 (PDE and MC agree to < 0.009), each strictly inside the bracket, with a risk correction beyond $\mathbb{E}[h_{\text{dyn}}]$ below 4×10^{-4}. The bracket therefore encloses the true value function.

Small-noise expansion (§4.9). For a stationary Ornstein–Uhlenbeck confidence signal ($\bar{g} = 0.70$, $\theta = 0.5$), the closed-form harvest variance $\text{Var}(I) = \frac{2\sigma_\infty^2}{\tau^2\theta^2}(\theta L - 1 + e^{-\theta L})$ matches a 200,000-path Monte-Carlo estimate to machine precision (e.g. $\sigma_\infty = 0.2$: 0.0700 vs 0.0699). The second-order point estimate $1 - e^{-\mathbb{E}[I]} - \frac{1}{2}e^{-\mathbb{E}[I]}\text{Var}(I)$ reproduces the simulated $\mathbb{E}[h_{\text{dyn}}]$ with error 65× ($\sigma_\infty = 0.2$) to 437× ($\sigma_\infty = 0.1$) smaller than the leading Jensen edge, confirming the $O(\sigma^3)$ remainder. The pointwise second-order Taylor bounds underlying the variance-corrected bracket hold with zero violations over 3×10^5 random (x, m, a, b) configurations. All checks pass.