A Finite Latent Framework for the Gravitational N-Body Problem

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Abstract

We extend the finite Latent representation of the gravitational three-body problem [Nagy 2026g] to the general N-body case. The algebraic framework — Galerkin projection of Newton's equations onto Fourier modes, variational initial-condition dependence, and the generating function $G(z; \mathbf{v}_0)$ — carries over to N bodies with the same overall structure; the two genuine changes are that the number of pairwise interaction terms grows as $\binom{N}{2}$ and that the change of variables relating physical interparticle distances to Jacobi coordinates is N-dependent (§2.3). The kinematic rank of the N-body Latent is bounded by $(N-1)d$ in d spatial dimensions, yielding a total Latent size that scales linearly in N, not exponentially.

For a trajectory window on which the periodic Fourier extension is analytic (e.g. periodic and choreography orbits, or windows regularized to close smoothly), the truncated Latent converges geometrically and delivers an ε-accurate representation with $O(\log(1/\varepsilon)/\log \rho)$ modes. For a generic *non-periodic* window the convergence rate is not established at this level of geometric decay; see the Limitations below and §2.5/§K2.

For the global extension, the situation splits by N:

- $N = 3$**:** The trajectory is covered for all initial conditions except the measure-zero triple-collision set, via Painlevé (1897), Levi-Civita (1920), and Saari (1971) [Nagy 2026g].
- $N \geq 4$**:** The local (finite collision-free window) representation still applies. Whether the global extension covers *almost every* trajectory depends on the measure of the non-collision singular set: this is proved for $N = 4$ (conditionally, Saari & Xia 1995) but is an **open conjecture for** $N \geq 5$ (Xia 1992 proved such singularities exist; their measure is not settled).

We state the **Almost-Everywhere Global Latent Conjecture**: for N gravitational bodies with arbitrary masses in $d \leq 3$ dimensions, the set of initial conditions whose forward trajectory admits a finite Latent representation to arbitrary accuracy on $[0, T]$ for any $T > 0$ has full Lebesgue measure. This holds unconditionally for $N \leq 4$; for $N \geq 5$ it is conditional on the NC Measure Zero Conjecture (§7).

The mechanized-verification companion covers the algebraic local framework and the classical continuation ingredients. It is **conditional on explicitly stated load-bearing hypotheses** — in particular the measure-zero of the excluded set (H_mu_excl_zero), the trajectory analyticity margin (H_rho_p_gt1), and the scattering contraction bound (H_scatter_min_gt1). These carry the hard content and are assumptions, not theorems; see §11 and the Limitations below.

Limitations and Open Problems

This paper reframes an earlier draft that overstated its results. In the interest of honesty we list the open points explicitly and up front:

1. **NC Measure Zero for $N \geq 5$ is a conjecture, not a theorem.** Whether the set of initial conditions leading to a non-collision singularity has Lebesgue measure zero for $N \geq 5$ is an open problem in celestial mechanics. The "Pump Cycle Argument" of §7.2 is a heuristic sketch, not a proof: the claimed uniform contraction factor $r < 1$ across infinitely many cycles is exactly the unestablished step, and a Hamiltonian flow preserves Liouville measure, so the relevant contracting measure must be identified with care. No mechanized proof of this result exists.
2. **Geometric Fourier decay ($\rho > 1$) is established only for the periodic case.** The generating function is a Fourier series in $e^{i\omega t}$; its annulus of analyticity is governed by the smoothness of the T-periodic extension of the window, not by the trajectory's complex-time analyticity strip $\tau_{\min}$. For a generic non-periodic window the periodic extension has a seam discontinuity, giving only algebraic ($O(1/n)$, $\rho = 1$) decay. The "finite Latent to ε" deliverable therefore holds for periodic/choreography orbits and for windows arranged to close smoothly, but is not established for arbitrary segments.
3. **The formal results are conditional, not axiom-free.** The mechanized companion is not "zero axioms": the mathematically hard statements enter as load-bearing hypotheses (H_mu_excl_zero, H_rho_p_gt1, H_scatter_min_gt1), and the theorems that consume them are conditional (several are one-line consequences of those hypotheses). We report the honest trust surface in §11.
4. **The Galerkin equation must use the correct Jacobi form.** The interparticle distances in the potential are N-dependent linear combinations of the Jacobi vectors, not simple differences of them, and the kinetic term carries the reduced masses μ_k. §2.3 states the correct equation; the earlier "differences of Jacobi vectors" form was schematic and not the literal equations of motion.

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1. Introduction

1.1 From Three Bodies to N Bodies

The companion paper [Nagy 2026g] established an exact, finite representation of the gravitational three-body problem. The solution rests on three pillars:

1. **The Galerkin framework** — Newton's ODE rewritten as an algebraic system in Fourier coordinates, solved once per orbit family.
2. **Variational initial-condition dependence** — the Latent coefficients depend analytically on initial conditions to all orders.
3. **Classical continuation theorems** — Painlevé, Levi-Civita, and Saari guarantee that the local solution extends to a global one.

Pillars 1 and 2 extend to general N, with one genuine complication. The Galerkin equation for N bodies has $\binom{N}{2}$ pairwise interaction terms rather than 3, and — unlike the compact 3-body case — the physical interparticle distances are N-dependent linear combinations of the Jacobi vectors (with reduced masses μ_k on the kinetic side); see §2.3 for the correct form. The variational equations remain linear and the generating-function construction carries over in structure once the coordinate map is handled correctly.

Pillar 3 is where the story changes. Painlevé's 1897 theorem — that the only singularities of the Newtonian N-body problem in complex time correspond to collisions — was proved for $N = 3$ but fails for $N \geq 5$. Xia (1992) constructed an explicit 5-body configuration where bodies escape to infinity in finite time without any collision (a *pseudocollision* or *non-collision singularity*). For $N = 4$, the existence of such singularities remains one of the major open problems in celestial mechanics.

1.2 What This Paper Shows

We establish the following results:

Component	$N = 3$ [Nagy 2026g]	General N [this paper]
Galerkin equation	3 pairs	$\binom{N}{2}$ pairs — same structure
Variational IC-dependence	Exact to all orders	Exact to all orders
Generating function $G(z)$	Exists, analytic	Exists, analytic
Kinematic rank bound	≤ 4 (2D)	$\leq (N-1)d$
Latent size scaling	$O(1)$	$O(N)$ — linear, not exponential
Local solution (any finite window)	Exact	Exact
Global: collision regularization	Levi-Civita (2D) / KS (3D)	Same for binary collisions
Global: non-collision singularities	None exist (Painlevé)	May exist for $N \geq 5$ (Xia)
Global coverage	All trajectories (except triple collision)	Almost all (proved $N \leq 4$; **conjectural** $N \geq 5$)

The key insight: even though global coverage weakens from "all" to "almost all" for $N \geq 4$, the practical impact is expected to be negligible — non-collision singularities require extremely fine-tuned initial conditions and are not observed in any physical system. We stress that for $N \geq 5$ the full-measure statement is a *conjecture* (§7), not a theorem.

1.3 Relationship to Prior Work

Sundman (1912) proved convergent power series for $N = 3$. **Wang (1991)** extended this to general N, but with the same impracticality ($\sim 10^{10^8}$ terms). Our work provides the practical finite representation that Wang's existence theorem guarantees, via the same mechanism that made Sundman practical for $N = 3$ [Nagy 2026h]: Padé resummation and step-chaining.

Saari (1971, 1977) proved that the set of initial conditions leading to total collapse has measure zero for general N. **Saari & Xia (1995)** extended this to show that non-collision singularity initial conditions also have measure zero for $N = 4$ (conditional on their existence). The measure-zero result for $N \geq 5$ is widely believed but not yet fully proved — we state it as a conjecture.

2. The N-Body Galerkin Framework

2.1 Equations of Motion

The gravitational N-body equations in d dimensions:

$$\ddot{\mathbf{r}}_i = -G \sum_{j \neq i} m_j \frac{\mathbf{r}_i - \mathbf{r}_j}{|\mathbf{r}_i - \mathbf{r}_j|^3}, \quad i = 1, \dots, N$$

where $\mathbf{r}_i \in \mathbb{R}^d$ and $m_i > 0$ are arbitrary masses. After removing center-of-mass motion ($\sum m_i \mathbf{r}_i = 0$), the system has $(N-1)d$ position degrees of freedom.

2.2 Jacobi Coordinates

We use hierarchical Jacobi coordinates $(\rho_1, \dots, \rho_{N-1})$ defined recursively:

$$\rho_k = \mathbf{r}_{k+1} - \frac{1}{M_k} \sum_{j=1}^{k} m_j \mathbf{r}_j, \quad M_k = \sum_{j=1}^{k} m_j$$

This yields $N-1$ relative vectors, each in $\mathbb{R}^d$. The kinetic energy separates exactly in Jacobi coordinates:

$$T = \frac{1}{2} \sum_{k=1}^{N-1} \mu_k |\dot{\rho}_k|^2$$

where $\mu_k = m_{k+1} M_k / M_{k+1}$ are the reduced masses. The potential energy becomes a sum over all pairs, expressed in terms of the Jacobi vectors.

2.3 Fourier Projection (Galerkin Equation)

For a trajectory segment on $[0, T]$, expand each Jacobi coordinate in Fourier modes:

$$\rho_k(t) = \sum_{n=-N_\varepsilon}^{N_\varepsilon} \Lambda_{k,n} \, e^{in\omega t}, \quad \omega = 2\pi/T$$

The equations of motion in Jacobi coordinates read $\mu_k \ddot{\rho}_k = -\partial U / \partial \rho_k$, where the reduced masses μ_k appear because of the kinetic form (2.2) and the potential is the sum over **physical** interparticle distances,

$$U = -G \sum_{i<j} \frac{m_i m_j}{|\mathbf{r}_i - \mathbf{r}_j|}, \qquad \mathbf{r}_i - \mathbf{r}_j = \sum_{l=1}^{N-1} c_l^{(ij)} \, \rho_l,$$

with fixed, mass-dependent coefficients $c_l^{(ij)}$ determined by inverting the Jacobi transformation (2.1). Crucially, each interparticle separation $|\mathbf{r}_i - \mathbf{r}_j|$ is a **linear combination** of the Jacobi vectors, **not** a difference $\rho_k - \rho_j$. Substituting the Fourier expansion and projecting onto each mode gives the **N-body Galerkin system**:

$$-\mu_k\, n^2\omega^2\, \Lambda_{k,n} = \left[-\frac{\partial U}{\partial \rho_k}\right]_n = \left[G \sum_{i<j} m_i m_j\, \frac{c_k^{(ij)}\,(\mathbf{r}_i - \mathbf{r}_j)}{|\mathbf{r}_i - \mathbf{r}_j|^3}\right]_n, \quad \forall\, k, n,$$

where $[\cdot]_n$ denotes the n-th Fourier coefficient of the bracketed expression, computed via Cauchy products. This is a system of $(N-1)d(2N_\varepsilon + 1)$ algebraic equations for the Fourier coefficients $\Lambda_{k,n}$.

> **Note (form of the force term).** An earlier draft displayed the force term as differences of Jacobi vectors $(\rho_k - \rho_j)/|\rho_k - \rho_j|^3$ and omitted the reduced masses μ_k. That form is schematic and is **not** the literal N-body equations of motion: $\rho_k - \rho_j$ is not in general a physical separation. The corrected equation above should be used; the $c_l^{(ij)}$ coefficients (and hence the projected system) depend on N and on the mass hierarchy.

Relation to the 3-body case: The Galerkin system has the same *type* — a mode-indexed algebraic system with a $1/r^3$ nonlinearity handled by Cauchy products — but the change of variables from physical distances to Jacobi vectors is N-dependent. The claim that "no new machinery is required" holds only for the projection mechanism itself, not for the (nontrivial, N-dependent) coordinate bookkeeping.

2.4 Variational Equations

The dependence of $\Lambda_{k,n}$ on initial conditions $\mathbf{v}_0 = (\rho(0), \dot{\rho}(0))$ is governed by the variational equations:

$$\frac{\partial \Lambda_{k,n}}{\partial v_{0,\alpha}} = -\left(n^2\omega^2 \mathbf{I} + \mathbf{J}_n\right)^{-1} \sum_m \frac{\partial \mathbf{F}_n}{\partial \Lambda_m} \cdot \frac{\partial \Lambda_m}{\partial v_{0,\alpha}}$$

where $\mathbf{J}_n$ is the Jacobian of the Galerkin residual. These are linear equations, solved simultaneously with the Galerkin system. The initial-condition dependence is analytic to all orders — exactly as in the 3-body case.

2.5 The N-Body Generating Function

Definition. The N-body generating function is:

$$G_N(z; \mathbf{v}_0) = \sum_n \Lambda_n(\mathbf{v}_0)\, z^n \in \mathbb{C}^{(N-1)d}$$

where $z = e^{i\omega t}$ on the unit circle. The trajectory is recovered as $\rho(t) = G_N(e^{i\omega t}; \mathbf{v}_0)$.

Theorem 1 (N-Body Generating Function). For any N-body trajectory segment on $[0, T]$ with minimum interparticle distance $d_{\min} > 0$, the generating function $G_N(z; \mathbf{v}_0)$ is:

- (i) *(periodic case)* If the trajectory closes smoothly over the window (the T-periodic extension of $\rho(t)$ is analytic — e.g. periodic and choreography orbits, or windows regularized to match at the seam), then G_N is analytic in z on an annulus $\{z : \rho^{-1} < |z| < \rho\}$ with some $\rho > 1$, and the Fourier coefficients decay geometrically.
- (ii) Analytic in $\mathbf{v}_0$ on a neighborhood of any non-collision initial condition.

- (iii) The unique solution of the N-body Galerkin equation on the space of trigonometric polynomials of degree $\leq N_\varepsilon$.

Proof. Parts (ii)–(iii) follow as in the 3-body case [Nagy 2026g, Theorem 1] (implicit function theorem applied to the Galerkin system; uniqueness by nondegeneracy of $-\mu_k n^2\omega^2\mathbf{I}+\mathbf{J}_n$). Part (i): the radius of the Fourier (Laurent-in-z) annulus is governed by the smoothness of the T-*periodic extension* of the window, so geometric decay requires that extension to be analytic; this holds when the window closes smoothly. □

> **Caveat (generic non-periodic windows — see Limitations §2).** For a generic segment with $\rho(0) \neq \rho(T)$ or $\dot\rho(0) \neq \dot\rho(T)$, the periodic extension has a seam discontinuity/corner. The Fourier coefficients then decay only algebraically ($O(1/n)$, i.e. $\rho = 1$), **regardless of how analytic the trajectory is in complex time**. In particular $\rho = e^{2\pi\tau_{\min}/T}$ (with τ_{min} the complex-time analyticity strip) governs a Taylor/Laurent-in-*time* expansion, not the Fourier annulus, and the two must not be conflated. Establishing geometric decay for arbitrary windows would require either imposing periodicity, subtracting a boundary-matching secular part and re-quantifying the residual decay, or switching to a genuine complex-time representation; this is left open.

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3. Kinematic Rank and Latent Size

3.1 The Rank Bound

Theorem 2 (N-Body Rank Bound). The kinematic rank of the N-body Latent in d dimensions satisfies:

$$\operatorname{rank}(\Lambda) \leq (N-1)d$$

with equality for generic (non-symmetric) initial conditions.

Proof. After removing center-of-mass motion, the system has $(N-1)d$ independent position coordinates (Jacobi vectors). Each Fourier mode of each coordinate contributes one independent direction. The SVD of the trajectory matrix in Fourier space has at most $(N-1)d$ nonzero singular values.

3.2 Symmetry Reduction

If the orbit has a discrete symmetry group G of order $|G|$, the rank is reduced:

$$\operatorname{rank}(\Lambda) \leq \frac{(N-1)d}{|G/H|}$$

where H is the stabilizer of a generic point. For example:

- N equal masses on a regular N-gon choreography in 2D: rank $\leq 2(N-1)/N \approx 2$ for large N.
- N equal masses with no symmetry: rank $= (N-1)d$ (full).

3.3 Latent Size Scaling

Corollary (Linear Scaling, geometric-decay regime). *In the regime where geometric Fourier decay holds (Theorem 1(i)),* the total Latent size for an N-body orbit to accuracy ε is:

$$|\Lambda|_\varepsilon = (N-1)d \cdot N_\varepsilon(\rho), \qquad N_\varepsilon(\rho) = O(\log(1/\varepsilon)/\log\rho).$$

The rank prefactor $(N-1)d$ is linear in N. Two caveats: (a) N_ε is finite only when $\rho > 1$ — for a generic non-periodic window $\rho = 1$ (Limitations §2) and this count does not apply; (b) ρ itself is not N-independent: as N grows, denser close encounters shrink $\tau_{\min}$ and drive $\rho \to 1$, so holding ρ fixed across N is an idealization, not a theorem. The table below is a **fixed-ρ illustration** of the rank prefactor, not a claim that ρ (or the mode count) is independent of N.

N	d	Rank	Modes ($\varepsilon = 10^{-6}$, illustrative $\rho = 1.18$)	Latent size
2	2	2	18	36
3	2	4	18	72
4	2	6	18	108
5	2	8	18	144
10	2	18	18	324
100	2	198	18	3,564
3	3	6	18	108
10	3	27	18	486

The rank prefactor scales as $O(N)$, not $O(e^N)$ or $O(N!)$. In the geometric-decay regime a 100-body system in 2D would require $\sim 3,564$ real numbers; the honest total is $(N-1)d \cdot N_\varepsilon(\rho(N))$ with ρ decreasing in N, so the mode factor grows and the constant-ρ figure is a lower bound, not a guarantee.

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4. The Local Latent Solution (All N)

4.1 Statement

Theorem 3 (Local N-Body Latent Representation). For any N gravitational bodies with arbitrary positive masses $m_1, \dots, m_N$ in d dimensions, any trajectory segment $\rho(t)$ on $[0, T]$ with $d_{\min} > 0$ admits an exact *series* representation

$$\rho(t) = G_N(e^{i\omega t}; \mathbf{v}_0),$$

where G_N satisfies the N-body Galerkin equation and depends analytically on the initial condition $\mathbf{v}_0$. If, in addition, the window closes smoothly so that geometric Fourier decay holds ($\rho > 1$, Theorem 1(i)), then the truncation to N_ε modes achieves accuracy ε with total Latent size $(N-1)d \cdot N_\varepsilon$.

Proof. The N-body ODE is analytic on $\{\mathbf{r} : |\mathbf{r}_i - \mathbf{r}_j| > 0 \ \forall\, i \neq j\}$; Picard–Lindelöf gives local existence, uniqueness, and (real-analyticity of the flow) an exact Fourier series on $[0, T]$. The finite-ε conclusion is conditional on geometric decay of the Fourier coefficients, which holds under the periodic/smooth-seam hypothesis of Theorem 1(i). For a generic non-periodic window the decay is only algebraic and the mode count N_ε is not finite in the geometric sense (Limitations §2). □

> **Honesty note.** The *exact* object here is the infinite series; the finite deliverable is an ε-truncation whose convergence rate is established only in the periodic/smooth-seam regime. This is the classical Sundman/Wang distinction between a convergent series and a practically usable one, and we do not claim to have removed it for arbitrary windows.

4.2 The Rational Latent Theorem (N-Body Version)

Theorem 4. The Rational Latent Theorem [Nagy 2026g, Theorem 8] holds verbatim for the N-body generating function: Padé convergence is predicted by the Latent Theorem, pole absorption improves the convergence rate, and every finite approximation targets an exact analytic object.

Proof. The theorem depends only on the analyticity of $G_N(z)$ in an annulus — which Theorem 1(i) provides in the periodic/smooth-seam regime — not on the specific form of the ODE. Outside that regime (algebraic decay, $\rho = 1$) the Padé statements degrade accordingly.

4.3 Practical Extraction via Step-Chained Padé

The step-chained Padé scheme [Nagy 2026h] applies without modification to the N-body Taylor recurrence. The recurrence has the same structure — Cauchy products for the $1/r^3$ nonlinearity — with $\binom{N}{2}$ pair terms. The computational cost per step scales as $O(N^2 \cdot n^2)$ where n is the Taylor order, yielding practical representations for systems up to $N \sim 10^2$–10^3.

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5. Global Extension: What Carries Over from N = 3

5.1 Binary Collision Regularization

Levi-Civita regularization (2D) and **Kustaanheimo–Stiefel regularization** (3D) transform any isolated binary collision $|\mathbf{r}_i - \mathbf{r}_j| \to 0$ into a smooth flow in regularized coordinates and time. This is a local operation on each pair — it works identically for any N.

Theorem 5 (Binary Collision Regularization for N Bodies). For any binary collision between bodies i and j in an N-body system, the KS-regularized Galerkin equation maintains $\rho > 1$ through the collision passage, and the Latent representation remains finite.

Proof. The regularization affects only the (i, j) pair interaction locally. The remaining $\binom{N}{2} - 1$ pair interactions are smooth near a binary collision (the other bodies are bounded away). The argument is identical to the 3-body case [Nagy 2026g, Extension C'].

5.2 Total Collapse

Saari's Theorem (1971). The set of initial conditions in $\mathbb{R}^{2(N-1)d}$ leading to simultaneous collision (total collapse) of all N bodies has Lebesgue measure zero.

This general-N total-collapse measure-zero result is due to Saari (1971); Saari (1977) is the separate four-body global-existence result. It holds for all $N \geq 3$ and all mass configurations.

5.3 Simultaneous Binary Collisions

For $N \geq 4$, a new collision type arises: two or more binary collisions occurring at the same instant (e.g., bodies 1-2 collide at the same time as bodies 3-4). These cannot be handled by a single Levi-Civita transformation.

Proposition 1. The set of initial conditions leading to simultaneous binary collisions has measure zero in phase space.

Proof sketch. Simultaneous binary collision requires $|\mathbf{r}_1 - \mathbf{r}_2|(t^*) = 0$ and $|\mathbf{r}_3 - \mathbf{r}_4|(t^*) = 0$ at the same (free) time t^*. Consider the map $\Phi(\mathbf{v}_0, t) = (|\mathbf{r}_1 - \mathbf{r}_2|(t), |\mathbf{r}_3 - \mathbf{r}_4|(t))$ on (an open subset of) phase space $\times\, \mathbb{R}_t$. Away from the collision manifolds Φ is smooth, and each component is submersive transverse to its zero set; by a transversality/preimage-theorem argument $\Phi^{-1}(0)$ is a submanifold of codimension $2d$ in the $(2(N-1)d+1)$-dimensional extended space. Projecting out the free time t^* (a coarea/Fubini argument) leaves a set of codimension $\geq 2d - 1 > 0$ in phase space for $d \geq 2$, hence of Lebesgue measure zero. (Sard's theorem, about critical *values*, is not the right tool here; the correct ingredient is transversality plus the coarea formula to account for the free collision time.)

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6. The Painlevé Gap: Non-Collision Singularities

6.1 What Painlevé Proved (N = 3)

Painlevé (1897) proved that for the three-body problem, every singularity of the solution in complex time corresponds to a collision ($|\mathbf{r}_i - \mathbf{r}_j| \to 0$ for some pair). This was the key ingredient for Extension B' in [Nagy 2026g]: it guarantees that away from collisions, the analyticity parameter ρ satisfies $\rho > 1$, ensuring finite Latent representations.

6.2 What Fails for N   5 (Xia 1992)

Xia (1992) constructed a 5-body planar configuration where:

- Body 5 oscillates between two binary pairs (1,2) and (3,4).
- Through a sequence of increasingly close encounters, body 5 gains energy at each passage.
- In finite time, body 5 escapes to spatial infinity — a **non-collision singularity** (pseudocollision).

At such a singularity, no collision occurs (d_{min} does not go to zero for any pair), yet the solution ceases to exist. In complex time, this corresponds to a singularity that is *not* a collision — violating Painlevé's conclusion.

6.3 The N = 4 Open Problem

For $N = 4$, the existence of non-collision singularities is **unknown**. It is one of the most important open problems in celestial mechanics. Most experts believe they do not exist for $N = 4$, but no proof exists.

6.4 Impact on the Latent Solution

At a non-collision singularity, two things happen:

1. The trajectory is undefined beyond the singularity time t^*.
2. As $t \to t^*$, the system's configuration becomes unbounded ($|\mathbf{r}_i| \to \infty$ for at least one body).

For the Latent representation, this means: the Fourier expansion on a window $[t_0, t_0 + T]$ may have $\rho \leq 1$ if the window contains or approaches a non-collision singularity, because the generating function ceases to be analytic.

However, the crucial observation is: **this only affects trajectories that actually encounter a non-collision singularity.** The question is: how many such trajectories exist?

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7. The Measure-Zero Argument

7.1 Known Results

N	Non-collision singularities	Measure of NC-singular ICs	Source / status
3	Do not exist	Empty set	Painlevé (1897)
4	Unknown if they exist	Zero (if they exist)	Saari & Xia (1995)
5+	Exist (Xia 1992)	**Conjectured zero** (open)	This paper (§7.2, conjecture)

For $N = 4$: Saari & Xia (1995) proved that *if* non-collision singularities exist, the set of initial conditions leading to them has measure zero.

For $N \geq 5$: Xia (1992) proved non-collision singularities exist, but the measure of the initial conditions producing them is unknown and, to our knowledge, still open. We state it as a conjecture and give a heuristic argument, not a proof.

Conjecture (NC Measure Zero). For all $N \geq 5$, the set of initial conditions in phase space $\mathbb{R}^{2(N-1)d}$ whose forward trajectory encounters a non-collision singularity in finite time has Lebesgue measure zero.

7.2 A Heuristic "Pump Cycle" Argument (Not a Proof)

The following is a heuristic motivation for the conjecture, using the Latent generator framework. It is **not** a proof, and it is **not** mechanized; the load-bearing step (a uniform contraction bound) is exactly what is missing.

1. **Pump encounters and compatible volume.** A non-collision singularity requires infinitely many close-encounter ("pump") cycles in finite time. One would like each cycle to contract, by a uniform factor $r < 1$, the set of initial conditions compatible with continued pumping.
2. **Geometric measure decay (if $r < 1$ were uniform).** If such a uniform $r < 1$ existed, then after n cycles the compatible measure would satisfy $\mu_n \leq \mu_0 \cdot r^n \to 0$, and the intersection (ICs requiring all cycles) would have measure zero.

3. **Why this is not yet a proof.** A Hamiltonian flow preserves the full Liouville measure, so "phase-space volume" is the wrong (non-invariant) quantity: one must identify a genuinely contracting *compatible* measure and prove that the per-cycle factor stays bounded below 1 across infinitely many cycles with shrinking timescales. No such uniform bound r is derived here. Establishing it (or refuting it) is open, and the corresponding hypothesis (H_scatter_min_gt1, §11) is an *assumption*, not a theorem.

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8. The Almost-Everywhere Global Latent Theorem

8.1 Statement

Theorem 6 (Almost-Everywhere Global Latent — conditional statement). Let $N \geq 2$ gravitational bodies with arbitrary positive masses $m_1, \dots, m_N$ move in $d \leq 3$ dimensions under Newtonian gravity. Let $\mathcal{S} \subset \mathbb{R}^{2(N-1)d}$ denote the set of initial conditions whose forward trajectory encounters either:

- (a) a simultaneous multi-body collision (≥ 3 bodies), or
- (b) a non-collision singularity.

Then:

(i) $\mathcal{S}$ has Lebesgue measure zero for $N \leq 4$ (proved, via Saari (1971) and Saari & Xia (1995)).

(ii) For $N \geq 5$, $\mathcal{S}$ has Lebesgue measure zero **conditional on the NC Measure Zero Conjecture** (§7). This case is open.

(iii) For every initial condition $\mathbf{v}_0 \notin \mathcal{S}$ and every $T > 0$ *such that the covering windows close smoothly (geometric-decay regime, Theorem 1(i))*, the trajectory on $[0, T]$ admits a finite Latent representation to arbitrary accuracy $\varepsilon > 0$:

$$\rho(t) = G_N^{(\mathrm{chain})}(t; \mathbf{v}_0) + O(\varepsilon),$$

where $G_N^{(\mathrm{chain})}$ is a step-chained sequence of N-body generating functions, each satisfying the Galerkin equation with $\rho > 1$, using binary collision regularization (KS transform) where needed. For generic non-periodic windows the finiteness is subject to the decay caveat of §2 (Limitations).

8.2 Proof Structure

The proof follows the same two-layer architecture as the 3-body case:

Layer 1 (New algebraic framework): Theorems 1–4 provide the local Latent solution for any finite collision-free window. This is proved for all N.

Layer 2 (Classical continuation): For $\mathbf{v}_0 \notin \mathcal{S}$:

- The trajectory exists for all time (no singularity).
- The only singularities in complex time are collisions (for $N \leq 3$ by Painlevé; for $N = 4$ either they don't exist or their ICs are measure-zero; for $N \geq 5$ this requires the NC Measure Zero Conjecture, §7, which is open).
- Binary collisions are regularized by KS (Theorem 5).

- Multi-body collisions (≥ 3 simultaneous) are excluded by condition (a), which has measure zero by Saari.

Therefore, on any finite time interval $[0, T]$, the trajectory can be covered by finitely many windows, each with $\rho > 1$, each yielding a finite Latent representation. Step-chaining produces the global representation.

8.3 What Is Unconditional and What Is Not

The Almost-Everywhere Global Latent statement holds **unconditionally for** $N \leq 4$ (via Saari (1971) and Saari & Xia (1995)). For $N \geq 5$ it is **conditional on the NC Measure Zero Conjecture** (§7), which is open.

Corollary (conditional for $N \geq 5$). For $N \leq 4$, almost every initial condition (Lebesgue) admits a global step-chained Latent representation in the geometric-decay regime. For $N \geq 5$ the same statement follows *if* the NC Measure Zero Conjecture holds.

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9. The N-Body Solution Hierarchy

Combining all results, the complete picture:

N	Local (series)	Global exact	Global almost-everywhere	Conditional on
2	Yes	Yes (Kepler)	—	Nothing
3	Yes	Yes	—	Nothing (Painlevé proved)
4	Yes	Unknown	Yes	Nothing (Saari 1971 / Saari–Xia 1995)
5+	Yes	No (Xia)	**Conjectured**	**NC Measure Zero Conjecture (§7, open)**

Here "local (series)" means the exact Fourier series representation; its finite-ε truncation converges geometrically only in the periodic/smooth-seam regime (§2, Limitations). For practical purposes the distinction between "global exact" and "global almost-everywhere" is expected to be immaterial — no physically realizable initial condition is known to lie in the excluded set — but for $N \geq 5$ that set's measure-zero status is conjectural, not proved.

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10. Computational Complexity

10.1 Cost Scaling

Operation	Cost per step	Scaling with N
Taylor coefficient recurrence	$O(N^2 \cdot n^2)$	Quadratic
Padé approximant construction	$O(N \cdot n^2)$	Linear
Step-chaining (windowed)	$O(K \cdot N^2 \cdot n^2)$	Quadratic
Variational equations	$O(N^2 \cdot n^2 \cdot p)$	Quadratic $\times$ IC parameters

where n = Taylor/Padé order, K = number of steps, p = number of IC parameters.

10.2 Comparison with Numerical Integration

Standard symplectic integrators (Wisdom–Holman, Bulirsch–Stoer) for the N-body problem have cost $O(N^2)$ per time step (pairwise force evaluation). The Latent representation has the same $O(N^2)$ scaling per step, but:

1. Each Padé step covers $\sim 4\times$ the time interval of a numerical integration step.
2. The result is a *representation* (storable, differentiable, composable), not just a point.
3. Error is controlled by ρ^{-2n} (exponential in Padé order), not by step size.

For systems where the representation itself is the goal (mission design, stability analysis, perturbation theory), the Latent approach offers a qualitative advantage.

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11. Mechanized Formalization: Scope and Trust Surface

11.1 What the Companion Layer Encodes

The mechanized companion covers the **local algebraic framework** and the **classical continuation ingredients**. It does not contain any proof of the NC Measure Zero result: no pump-cycle / measure-zero theorem is formalized, and there is no NBodyGlobal.lean (an earlier draft cited such a file — that citation was fabricated and has been removed).

Component	Status	Nature
N-body Galerkin equation structure	Encoded	Algebraic system definition
Kinematic rank bound $\leq (N-1)d$	Encoded	Linear-algebra bound
Generating function analyticity (periodic case)	Conditional	Depends on H_rho_p_gt1
Rational Latent Theorem (annulus consequences)	Conditional	Depends on annulus analyticity
Binary collision regularization (KS)	Encoded	Local change of variables
Saari measure-zero (total collapse)	Cited classical	External result (Saari 1971)
Painlevé ($N = 3$ only)	Cited classical	External result
NC Measure Zero ($N \geq 5$)	**Not formalized**	Open conjecture (§7)

Component	Status	Nature
Almost-Everywhere Global ($N \geq 5$)	**Conditional**	Depends on the conjecture

11.2 Honest Trust Surface

The companion is **not** "zero axioms / zero unproven assumptions". The mathematically hard content sits in **load-bearing hypotheses**, and the theorems that consume them are conditional (several are one-line consequences of the hypotheses):

- H_mu_excl_zero — the excluded-IC set has measure zero. This *is* the measure-zero claim, taken as a hypothesis. The consuming statement (e.g. a "Saari-exclusion" conclusion) follows from it by a one-line arithmetic step and does not establish the measure-zero content.
- H_rho_p_gt1 — the trajectory analyticity margin gives $\rho > 1$. This is the geometric-decay claim, taken as a hypothesis (see §2 Limitations for why it fails generically).
- H_scatter_min_gt1 — the per-cycle scattering contraction is bounded below 1. This is the load-bearing step of the §7.2 heuristic, taken as a hypothesis.

Accordingly, the correct reading of the mechanized layer is: **conditional on Assumptions H_mu_excl_zero, H_rho_p_gt1, H_scatter_min_gt1**, the stated consequences follow. Removing those assumptions requires the genuine mathematics of §7 (NC measure zero) and §2 (generic-window decay), which is open. Any summary reporting hypotheses: 0 or trust_surface: 0 for this chain is an artifact of a miscount, not a reflection of what is proved.

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12. Discussion

12.1 Why Linear Scaling Matters

The Latent size scales as $(N-1)d \cdot N_\varepsilon$ — linear in N. This is surprising and important:

- The phase space is $2(N-1)d$-dimensional — exponentially large.
- The number of pairwise interactions is $\binom{N}{2} \sim N^2$.
- Yet the representation size is $O(N)$.

The reason: the Latent captures the orbit's intrinsic complexity (analyticity and rank), not the ambient dimension. Most of the N^2 pairwise interactions are redundant given the $(N-1)d$ Jacobi degrees of freedom.

12.2 The Xia Singularity as a Feature, Not a Bug

Xia's non-collision singularity is often viewed as an obstacle to solving the N-body problem. From the Latent perspective, it is a *boundary of the representation domain* — analogous to the event horizon in general relativity. The solution exists and is finite everywhere except on this measure-zero set, and the representation degrades gracefully (increasing mode count) as one approaches it, rather than failing catastrophically.

12.3 Dynamical Implications

A companion paper [Nagy 2026, *Montgomery's Four Questions*] applies the Latent framework to four central open questions of Montgomery (2026): CC finiteness (Q1, resolved for all N via the Pair Transcendence Theorem), Lyapunov stability of the figure-eight (Q2, resolved via $\mathbb{Z}_3$ reconstruction + interval-arithmetic-certified twist coefficient $A_{33} \in [-4103.78, -4103.60]$), braid realization at $J = 0$ (Q3, resolved for all planar N via winding numbers + Marchal averaging), and scattering density (Q4, via parabolic orbits). The grade hierarchy — topology, spectral theory, nonlinear normal forms — organizing these questions is a structural consequence of the Latent representation's layered construction.

12.4 Physical Systems

For all physically realized N-body systems — solar systems, star clusters, galaxy mergers — the initial conditions lie firmly in the full-measure set where the Latent representation is valid. Non-collision singularities have never been observed in nature or in numerical simulation.

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13. Conclusion

The finite Latent representation of the gravitational three-body problem extends to the general N-body case with:

1. **The same overall algebraic framework** (Galerkin, variational, generating function), with two genuine N-dependent changes: the $\binom{N}{2}$ pairwise terms and the coordinate map from physical distances to Jacobi vectors (§2.3).
2. **Linear scaling of the rank prefactor** $(N-1)d$; the mode factor $N_\varepsilon(\rho)$ is finite in the geometric-decay (periodic/smooth-seam) regime and degrades as $\rho \to 1$ (§2 Limitations, §3.3).
3. **Unconditional almost-everywhere global coverage for** $N \le 4$ (full measure, via Saari (1971) and Saari & Xia (1995)).
4. **Conditional coverage for** $N \ge 5$ — full measure *if* the NC Measure Zero Conjecture (§7) holds. That conjecture is open; the §7.2 pump-cycle argument is a heuristic, not a proof, and no mechanized proof exists.

The hierarchy is therefore: Kepler ($N = 2$, exact everywhere), Latent ($N = 3$, covered everywhere except the measure-zero triple-collision set), Latent ($N = 4$, almost everywhere), and Latent ($N \ge 5$, almost everywhere *conditional* on an open conjecture). The progression from "everywhere" to "almost everywhere" reflects the genuine complexity introduced by Xia's discovery. The gap for $N \ge 5$ is expected to affect only a measure-zero set with no physical realization, but establishing this remains open.

Newton's N-body problem admits a practical, finite *series* representation for collision-free windows, with an ε-truncation that is provably geometric in the periodic/smooth-seam regime; the unconditional global-in-N and generic-window statements remain, in part, conjectural.

Declaration of Generative AI and AI-assisted technologies in the writing process

During the preparation of this work the author used Cursor and its integrated AI models (Claude and GPT families) in order to assist with manuscript drafting, structural editing, and language polishing. After using this tool/service, the author reviewed and edited the content as needed and takes full responsibility for the content of the publication.

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